A Proof of Dirichlet's Theorem on Primes in Arithmetic Progressions

Andrew Droll[1,*]

[1]*School of Mathematics and Statistics, Carleton University*

(Dated: January 5, 2007)

[*]Electronic address: `adroll@connect.carleton.ca`

I. INTRODUCTION

In this report we review general properties of Dirichlet characters and L-functions, and use these to prove Dirichlet's theorem that in any arithmetic progression $(ba+m)_{b=0}^{\infty}$ with $(a,m)=1$ there are infinitely many primes. This result was first proved in 1837 by J. Dirichlet. Euler had previously conjectured that there are infinitely many primes congruent to 1 modulo any nonzero integer. Dirichlet's theorem itself was first conjectured by Gauss.

We follow the general exposition in [1, Ch.16], elaborating and reordering at every stage. Section II begins by reviewing the properties of complex characters on groups and Dirichlet characters modulo m. In this section we also define Dirichlet L-functions. Our proof of Dirichlet's theorem relies heavily on the properties of L-functions. This being the case, Section III examines elementary properties of Dirichlet L-functions in some detail. We prove the existence of Euler product decompositions, and specify a branch of the logarithm of particular use for our proof. Additionally, we introduce the Riemann zeta function and several of its properties. Dirichlet's theorem is stated without proof in Section IV, and the existence of analytic continuations for Dirichlet L-functions is proved. In Section V, we approach the most difficult part of the proof of Dirichlet's theorem, involving the boundedness of L-functions and their logarithms. We conclude our proof of Dirichlet's theorem in Section VI.

Our presentation frequently takes advantage of elementary results from group theory, real analysis, number theory, and complex analysis. References on these topics can be found in the bibliography.

Notation: Wherever we work with abstract groups, we write the group operation multiplicatively.

$\boldsymbol{C}$, $\boldsymbol{R}$, $\boldsymbol{Z}$, and $\boldsymbol{N}$ denote the complex numbers, real numbers, integers, and natural numbers (i.e. positive integers), respectively.

$\boldsymbol{T}$ denotes the unit circle in the complex plane.

For $m \in \boldsymbol{Z}$, $\boldsymbol{Z}_m$ denotes the integers modulo m.

For a monoid M, $U(M)$ denotes the group of units of M.

For $z \in \boldsymbol{C}$, $\overline{z}$ denotes the complex conjugate of z.

For $z \in \boldsymbol{C}$, $\Re(z)$ denotes the real part of z, and $\Im(z)$ denotes the imaginary part of z.

ϕ denotes the Euler phi function, i.e. $\phi(m) = |U(\boldsymbol{Z}_m)|$ for all $m \in \boldsymbol{Z}\backslash\{0\}$.
If A is a finite group and $g \in A$ then $|g|$ denotes the order of g.
For a set S, $|S|$ denotes the cardinality of S.
Log denotes the principal branch of the complex natural logarithm, and log denotes the real natural logarithm.

II. PRELIMINARIES

We will introduce some fundamental definitions and properties in this section. These will be important in the proof of Dirichlet's theorem to follow. Throughout this section, A is a finite abelian group.

Definition 1.1. Any group homomorphism $\chi : A \to \boldsymbol{T}$ is called a <u>character</u> on A. We denote the set of all characters on a group A by $\hat{A}$.

Since A and $\boldsymbol{T}$ are abelian groups, by an elementary theorem in group thery, $\hat{A}$ is also an abelian group under the multiplication defined by $(\chi\psi)(a) = \chi(a)\psi(a)$ whenever $\chi, \psi \in \hat{A},\ a \in A$.

Definition 1.2. Let χ_0 denote the trivial character on A, that is, $\chi_0(a) = 1$ for all $a \in A$.

We take note of the following theorem from group theory, which we will need to prove some properties of characters. A proof can be found in Chapter 7 of [2].

Theorem 1. A finite abelian group can be expressed as a direct product of cyclic subgroups.

We can recast Theorem 1 in the following way: There exist $g_1, ..., g_n \in A$ such that every $g \in A$ can be expressed in the form $g_1^{f_1}...g_n^{f_n}$ for some unique $f_1, ..., f_n \in \boldsymbol{Z}$ with

$0 \leq f_i < |g_i|$ for each i.

Lemma 1.1. The group of characters $\hat{A}$ is isomorphic to A.

Proof: Take $g_1, ..., g_n \in A$ such that every $g \in A$ can be expressed in the form $g_1^{f_1}...g_n^{f_n}$ for some unique $f_1, ..., f_n \in \boldsymbol{Z}$ with $0 \leq f_i < |g_i|, 1 \leq i \leq n$, and let $e_i = |g_i|$ for each i.

It is clear that any group homomorphism of A is determined completely by its effect on $g_1, ..., g_n$. Let $\chi \in \hat{A}$. Since $g_j^{e_j} = e$ for each j (where e is the identity element of A), we get $\chi(g_j)^{e_j} = 1$, so we must have $\chi(g_j) = e^{2\pi i h_j/e_j}$, where $0 \leq h_j < e_j$. For each j, $1 \leq j \leq n$, define a character $\chi_j \in \hat{A}$ by $\chi_j(g_j) = e^{2\pi i/e_j}$ and $\chi_j(g_q) = 1$ when $q \neq j$. The character χ_j is well-defined; Indeed, if $g_1^{a_1}...g_n^{a_n} = a = b = g_1^{b_1}...g_n^{b_n}$ then we have $ab^{-1} = e$, so that $b_k \equiv a_k \pmod{e_k}$ for each k. Consequently, for each k we have some $m_k \in \boldsymbol{Z}$ with $a_k = b_k + m_k e_k$, giving $\chi_j(a) = e^{2a_j\pi i/e_j} = e^{2(b_j+m_k e_k)\pi i/e_j} = e^{2b_j\pi i/e_j} = \chi_j(b)$. By the discussion above, every character χ can be expressed as a product $\chi = \chi_1^{m_1}...\chi_n^{m_n}$, where for each j, $m_j \in \boldsymbol{N}$ is unique and $0 \leq m_j < e_j$. Further, all such products are characters because $\hat{A}$ is a group.

We now define $f : \hat{A} \to A$ by $f(\chi) = f(\chi_1^{m_1}...\chi_n^{m_n}) = g_1^{m_1}...g_n^{m_n}$. It is clear from the discussion above that f is a bijection. Furthermore, if $\chi' = \chi_1^{k_1}...\chi_n^{k_n}$, $\chi = \chi_1^{m_1}...\chi_n^{m_n}$ then $f(\chi'\chi) = f(\chi_1^{k_1+m_1}...\chi_n^{k_n+m_n}) = g_1^{k_1+m_1}...g_n^{k_n+m_n} = (g_1^{k_1}...g_n^{k_n})(g_1^{m_1}...g_n^{m_n}) = f(\chi')f(\chi)$, so f is a homomorphism. This shows that $\hat{A}$ is isomorphic to A, as desired. □

Theorem 2.

(a) If $\chi, \psi \in \hat{A}$ then $\sum_{a\in A} \chi(a)\overline{\psi(a)} = |A|\delta(\chi, \psi)$, where $\delta(\chi, \psi) = 1$ if $\psi = \chi$ and $\delta(\chi, \psi) = 0$ otherwise.

(b) If $a, b \in A$ then $\sum_{\chi\in\hat{A}} \chi(a)\overline{\chi(b)} = |A|\delta(a, b)$ where $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ otherwise.

Proof: Note that if $\chi \in \hat{A}$, $a \in A$, then $\overline{\chi(a)}\chi(a) = 1$ since $\chi(a)$ is a root of unity, so that $\overline{\chi(a)} = \chi(a)^{-1} = \chi(a^{-1}) = \chi^{-1}(a)$.

We first prove part (a) of the proposition. We can write

$$\sum_{a\in A} \chi(a)\overline{\psi(a)} = \sum_{a\in A} \chi(a)\psi^{-1}(a) = \sum_{a\in A} (\chi\psi^{-1})(a).$$

If $\chi = \psi$ then $\chi\psi^{-1}$ is the trivial character χ_0. Otherwise, $\chi\psi^{-1}$ is some nontrivial character. Define $\chi\psi^{-1} = \eta$. If $\chi \neq \psi$, then there exists $b \in A$ such that $\eta(b) \neq 1$. Since A is a group, for every $b, c \in A$ there exists some unique $a \in A$ with $ba = c$, and thus

$$\sum_{a\in A} \eta(a) \quad = \sum_{a\in A} \eta(ab) = \sum_{a\in A} \eta(a)\eta(b) = \eta(b) \sum_{a\in A} \eta(a),$$

which implies

$$(\eta(b) - 1) \sum_{a\in A} \eta(a) = 0.$$

However by assumption $\eta(b) \neq 1$, and we must have $0 = \sum_{a\in A} \eta(a) = \sum_{a\in A} \chi(a)\overline{\psi(a)}$. If $\psi = \chi$ then $\eta(a) = 1 \; \forall a \in A$ so indeed,

$$|A| = \sum_{a\in A} \eta(a) = \sum_{a\in A} \chi(a)\overline{\psi(a)}.$$

This proves (a).

We now prove part (b). Write

$$\sum_{\chi\in\hat{A}} \chi(a)\overline{\chi(b)} = \sum_{\chi\in\hat{A}} \chi(a)\chi(b^{-1}) = \sum_{\chi\in\hat{A}} \chi(ab^{-1}).$$

If $a \neq b$ then $c = ab^{-1} \neq e$ (the identity of A). As in the proof of Lemma 1.1, take $g_1, ..., g_n \in A$, $e_1, ..., e_n \in \boldsymbol{N}$ such that every $g \in A$ can be expressed in the form $g_1^{f_1}...g_n^{f_n}$ for some unique $f_1, ..., f_n \in \boldsymbol{Z}$ with $0 \leq f_i < e_i \; \forall i, 1 \leq i \leq n$. Further, define χ_i for $1 \leq i \leq n$ in the same way as in the proof of Lemma 1.1. Because $c \neq e$, we see that if $c = g_1^{c_1}...g_n^{c_n}$ (with $0 \leq c_i < e_i$ for each i), at least one of $c_1, ...c_n$ is not zero, say $c_j \neq 0$. Then $\chi_j(c) = \chi_j(g_1^{c_1}...g_n^{c_n}) = \chi_j(g_j^{c_j}) = e^{2\pi i c_j/j} \neq 1$.

Since $\hat{A}$ is a group, for every $\psi, \eta \in \hat{A}$ there exists some unique $\chi \in \hat{A}$ with $\psi\chi = \eta$, and thus

$$\sum_{\chi\in\hat{A}} \chi(ab^{-1}) \quad = \sum_{\chi\in\hat{A}} \chi(c) = \sum_{\chi\in\hat{A}} (\chi_j\chi)(c) = \sum_{\chi\in\hat{A}} \chi_j(c)\chi(c) = \chi_j(c) \sum_{\chi\in\hat{A}} \chi(c),$$

which implies

$$(\chi_j(c) - 1) \sum_{\chi\in\hat{A}} \chi(c) = 0.$$

By construction $\chi_j(c) \neq 1$, and we must have $\sum_{\chi\in\hat{A}} \chi(c) = 0$. Thus for $a \neq b$ we have $\sum_{\chi\in\hat{A}} \chi(a)\overline{\chi(b)} = 0$. On the other hand, if $a = b$ then $ab^{-1} = e$, and $1 = \chi(ab^{-1}) = \chi(a)\overline{\chi(b)}$ for all $\chi \in \hat{A}$. We get $|A| = |\hat{A}| = \sum_{\chi\in\hat{A}} \chi(ab^{-1}) = \sum_{\chi\in\hat{A}} \chi(a)\overline{\chi(b)}$ (using Lemma 1.1).

These two arguments prove (b). □

Definition 1.3. Let m be a positive integer, and let χ' be a character on $U(\boldsymbol{Z}_m)$. Define $\chi : \boldsymbol{N} \to \boldsymbol{C}$ by $\chi(n) = \chi'(n + m\boldsymbol{Z})$ if $(n, m) = 1$ and $\chi(n) = 0$ otherwise. Any function χ constructed in this manner is referred to as a Dirichlet character modulo m, and χ' is called the character associated with χ. We denote the set of all Dirichlet characters modulo m by Δ_m.

For the remainder of this section, let m be a positive integer, and let $\chi \in \Delta_m$. Furthermore, let $C_{>1}$ denote the set $\{s \in \boldsymbol{C} \mid \Re(s) > 1\}$.

Lemma 1.2. Let $\chi, \psi \in \Delta_m$, and let χ', ψ' be the characters associated with χ and ψ respectively. Then $\psi = \chi$ if and only if $\psi' = \chi'$. Further, there is a bijection between Δ_m and the set of characters on $U(\boldsymbol{Z}_m)$.

Proof: Suppose $\psi = \chi$, so that whenever $(a, m) = 1$ we have $\psi'(a + m\boldsymbol{Z}) = \psi(a) = \chi(a) = \chi'(a + m\boldsymbol{Z})$. By an elementary result, $(a, m) = 1 \iff a + m\boldsymbol{Z} \in U(\boldsymbol{Z}_m)$. This proves one implication.

Conversely, suppose $\psi' = \chi'$. By definition, when $(a, m) \neq 1$ we have $\psi(a) = 0 = \chi(a)$. When $(a, m) = 1$ we find $\psi(a) = \psi'(a + m\boldsymbol{Z}) = \chi'(a + m\boldsymbol{Z}) = \chi(a)$, and so $\psi = \chi$.

Finally, let $\hat{U}$ denote the set of characters on $U(\boldsymbol{Z}_m)$ and define $f : \hat{U} \to \Delta_m$ in the following way: for every $\chi' \in \hat{U}$, define $f(\chi')$ to be the Dirichlet character $\chi \in \Delta_m$ with $\chi(a) = \chi'(a + m\boldsymbol{Z})$ whenever $(a, m) = 1$. By the arguments above, f is injective. From the definition of a Dirichlet character modulo m, f is clearly surjective. □

Corollary 1.2.1.
(a) If $\chi, \psi \in \Delta_m$, then

$$\sum_{a=0}^{m-1} \chi(a)\overline{\psi(a)} = \phi(m)\delta(\chi, \psi),$$

where δ is defined as in Proposition 1.1.
(b) If $a, b \in \boldsymbol{Z}$ with $(a, m) = (b, m) = 1$ then

$$\sum_{\chi \in \Delta_m} \chi(a)\overline{\chi(b)} = \phi(m)\delta'(a, b),$$

where δ' is defined by $\delta'(a,b) = 1$ if $a \equiv b \pmod{m}$ and $\delta'(a,b) = 0$ otherwise.

Proof: For part (a), note that when $(a,m) \neq 1$, by definition $\chi(a) = \psi(a) = 0$. When $(a,m) = 1$, again by definition, $\chi(a) = \chi'(a + m\boldsymbol{Z})$ and $\psi(a) = \psi'(a + m\boldsymbol{Z})$ for some characters χ' and ψ' on $U(\boldsymbol{Z}_m)$. By elementary results, $(a,m) = 1 \iff a + m\boldsymbol{Z} \in U(\boldsymbol{Z}_m)$. Thus we can rewrite the sum in (a) as $\sum_{a \in U(\boldsymbol{Z}_m)} \chi'(a)\overline{\psi'(a)}$. By Proposition 1.1, this sum is equal to $|U(\boldsymbol{Z}_m)|\delta(\chi', \psi') = |U(\boldsymbol{Z}_m)|\delta(\chi, \psi)$, and since $\phi(m) = |U(\boldsymbol{Z}_m)|$ by definition, this finishes the proof.

For (b), note that since $(a,m) = (b,m) = 1$, for each Dirichlet character $\chi \in \Delta_m$, we have $\chi(a) = \chi'(a + m\boldsymbol{Z})$ and $\chi(b) = \chi'(b + m\boldsymbol{Z})$ for some character χ' on $U(\boldsymbol{Z}_m)$. Using this in conjunction with Lemma 1.2, we see that we can write the sum over Dirichlet characters $\sum_{\chi \in \Delta_m} \chi(a)\overline{\chi(b)}$ as $\sum_{\chi' \in \widehat{U(\boldsymbol{Z}_m)}} \chi'(a + m\boldsymbol{Z})\overline{\chi'(b + m\boldsymbol{Z})}$. By Proposition 1.2, this sum is equal to $|U(\boldsymbol{Z}_m)|\delta(a + m\boldsymbol{Z}, b + m\boldsymbol{Z})$. Now (b) follows from the argument above if we recall the definition $\phi(m) = |U(\boldsymbol{Z}_m)|$ and that $a + m\boldsymbol{Z} = b + m\boldsymbol{Z}$ if and only if $a \equiv b \pmod{m}$. □

Lemma 1.3. For all $s, t \in \boldsymbol{N}$, $\chi(st) = \chi(s)\chi(t)$.

Proof: First suppose $(s,m) = (t,m) = 1$. Clearly then $(st,m) = 1$, and so by definition we have $\chi(st) = \chi'(st + m\boldsymbol{Z})$, where χ' is some character on $U(\boldsymbol{Z}_m)$. Since χ' is a homomorphism, this gives $\chi(st) = \chi'(st + m\boldsymbol{Z}) = \chi'((s + m\boldsymbol{Z})(t + m\boldsymbol{Z})) = \chi'(s + m\boldsymbol{Z})\chi'(t + m\boldsymbol{Z}) = \chi(s)\chi(t)$. Next, suppose that $(s,m) \neq 1$ or $(t,m) \neq 1$. By definition $\chi(s)\chi(t) = 0$. Clearly, we have $(st, m) \neq 1$, and so $\chi(st) = 0$ as well. □

Lemma 1.4. For all $n, k \in \boldsymbol{N}$, $\chi(n + km) = \chi(n)$.

Proof: Let χ' be the character associated with χ, as in Definition 1.3. If $(n,m) \neq 1$ then by definition $\chi(n) = 0$. Clearly, in this case we have $(n+m, m) \neq 1$, and so $\chi(n+m) = 0$ as well. On the other hand, if $(n,m) = 1$ then by definition $\chi(n) = \chi'(n + m\boldsymbol{Z})$. In this case we have $(n+m, m) = 1$, so $\chi(n+m) = \chi'((n+m) + m\boldsymbol{Z})$. However, $(n+m) + m\boldsymbol{Z} = n + m\boldsymbol{Z}$, and we get $\chi(n+m) = \chi'((n+m) + m\boldsymbol{Z}) = \chi'(n + m\boldsymbol{Z}) = \chi(n)$.

Thus, we have found $\chi(n + m) = \chi(n)$ for all positive integers n. Suppose $\chi(n + km) = \chi(n)$ for some positive integers k, n. Then $n + km$ is a positive integer, and so by the result we have just proved, $\chi(n + (k+1)m) = \chi(n + km + m) = \chi(n + km) = \chi(n)$. By induction, we conclude that $\chi(n + km) = \chi(n)$ for all positive integers n and k. □

We are now ready to define Dirichlet L-functions. Before stating the definition, we will need to prove the convergence of series of a certain form. In the discussion that follows we use the principal branch of the complex logarithm, so that $z^x = e^{x\mathrm{Log}(z)}$ for $z, x \in \boldsymbol{C}$. Note that if z is a positive real number then $\mathrm{Log}(z) \in \boldsymbol{R}$, from which it follows that

$$|z^x| \quad = |e^{(\Re(x)+i\Im(x))\mathrm{Log}(z)}| = e^{\Re(x)\mathrm{Log}(z)} = z^{\Re(x)}.$$

Proposition 1.1. Let $f : \boldsymbol{N} \to \boldsymbol{C}$ be any function with $|f(n)| \leq 1$ for every positive integer n. Then the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ is absolutely convergent for all $s \in \boldsymbol{C}_{>1}$ and uniformly convergent on $\{s \in \boldsymbol{C} \mid \Re(s) > 1 + \delta\}$ whenever $\delta > 0$.

Proof: Let $s \in \boldsymbol{C}$ with $\sigma = \Re(s) > 1$. Then by the comment above, $|f(n)n^{-s}| = |f(n)||n^{-s}| \leq |n^{-s}| = n^{-\sigma}$ for all positive integers n. Now, $n \mapsto n^{-\sigma}$ is a monotone decreasing continuous positive function of n, and so by the integral test for series convergence (see [3, Ch.19]), $\sum_{n=1}^{\infty} n^{-\sigma}$ converges if and only if $\int_1^{\infty} n^{-\sigma} dn$ converges. Since $\sigma > 1$, $\lim_{n\to\infty} n^{-\sigma+1} = 0$, and so

$$\int_1^{\infty} n^{-\sigma} dn \quad = \left[\frac{n^{-\sigma+1}}{-\sigma+1}\right]_1^{\infty} = \left(\lim_{n\to\infty} \frac{n^{-\sigma+1}}{-\sigma+1}\right) - \frac{1}{-\sigma+1} = \frac{1}{\sigma-1}.$$

The integral converges, so $\sum_{n=1}^{\infty} n^{\sigma}$ converges as well. Thus $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely whenever $\Re(s) > 1$ as we wanted.

Let $\delta > 0$. Then $|n^{-s}| < n^{-(1+\delta)}$ for all s with $\Re(s) > 1+\delta$. Here $\sum_{n=1}^{\infty} n^{-(1+\delta)}$ converges by the argument above, and so $\sum_{n=1}^{\infty} n^{-s}$ converges uniformly for all $s > 1 + \delta$ by the Weierstrass M-test [3, Ch.20]. □

Proposition 1.1 implies that $\sum_{n=1}^{\infty} \chi(n)n^{-s}$ converges absolutely on $\boldsymbol{C}_{>1}$ and uniformly on $\{s \in \boldsymbol{C} \mid \Re(s) > 1 + \delta\}$ whenever $\delta > 0$. This invites the following definition, which will conclude the section.

Definition 1.4. Let $L_\chi : \boldsymbol{C}_{>1} \to \boldsymbol{C}$ be defined by $L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$. We refer to L_χ as the the Dirichlet L-function associated with χ.

By Proposition 1.1, the series for L_χ converges everywhere on $\boldsymbol{C}_{>1}$.

It is important to note that our notation for L-functions is not standard. In the literature, the L-function associated with a Dirichlet character χ is typically denoted by $L(s,\chi)$. This standard notation is not in accord with the way in which functions are handled in modern mathematics. Our modified notation eliminates the problem.

III. PROPERTIES OF DIRICHLET L-FUNCTIONS

Here we present some general results on L-functions and the Riemann zeta function which will be necessary for the proof of Dirichlet's theorem. Let m be a positive integer throughout this section, and χ be a Dirichlet character modulo m.

Proposition 2.1. $L_\chi(s)$ is analytic for all $s \in \boldsymbol{C}_{>1}$.

Proof: By Proposition 1.1, $L_\chi(s) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ is uniformly convergent on $\{s \in \boldsymbol{C} | \Re(s) > 1 + \delta\}$ whenever $\delta > 0$. Clearly $\chi(n) n^{-s}$ is analytic for $s \in \boldsymbol{C}_{>1}$, so uniform convergence ensures that L_χ is analytic as well (see [4, Ch.5]). □

Theorem 3 [Euler Decomposition]. Let f be a multiplicative function on the positive integers, i.e. for all $m, n \in \boldsymbol{N}$, we have $f(mn) = f(m)f(n)$. Assume also that $|f(n)| \leq 1$ for all $n \in \boldsymbol{N}$. Let $s \in \boldsymbol{C}$ with $\Re(s) > 1$. Then

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \prod_p (1 - f(p)p^{-s})^{-1},$$

where the product is over all primes p.

Proof: By Proposition 1.1, $\sum_{n=1}^{\infty} f(n) n^{-s}$ converges absolutely. Let p be any prime. Since $\Re(s) > 1$, we have $|f(p)p^{-s}| = |f(p)||p^{-s}| \leq |p^{-s}| = |p^{-\Re(s)}| \leq 1$ and so using the usual expression for a geometric series we see that $\sum_{m=0}^{\infty} f(p)^m p^{-ms}$ converges absolutely to $(1 - f(p)p^{-s})^{-1}$. Let N be a positive integer and consider

$$\prod_{p \leq N} (1 - f(p)p^{-s})^{-1} = \left(\sum_{m_1=0}^{\infty} f(2)^{m_1} 2^{-m_1 s} \right) \left(\sum_{m_2=0}^{\infty} f(3)^{m_2} 3^{-m_2 s} \right) \cdots \left(\sum_{m_k=0}^{\infty} f(p_k)^m p_k^{-m_k s} \right),$$

where here p_i denotes the ith prime and p_k is the largest prime less than or equal to N. Each factor on the right side of the equation above is an absolutely convergent series, and we can rewrite the product as a multiple summation,

$$\begin{aligned}\prod_{p\leq N}(1-f(p)p^{-s})^{-1} &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty} f(2)^{m_1}2^{-m_1 s}\cdots f(p_k)^{m_k}p_k^{-m_k s}\\ &= \sum_{m_1=0}^{\infty}\cdots\sum_{m_k=0}^{\infty} f(2^{m_1}\dots p_k^{m_k})(2^{m_1}\dots p_k^{m_k})^{-s}.\end{aligned}$$

For fixed $m_1,\cdots,m_k$, $n = 2^{m_1}\cdots p_k^{m_k}$ is an integer. The set of prime factors of n is some subset of $\{2,3,\dots,p_k\}$. By choosing nonnegative integers $m_1,\dots,m_k$ appropriately we can represent any integer whose prime factors are a subset of $\{2,3,\dots,p_k\}$ as $2^{m_1}\dots p_k^{m_k}$. Now, if j is a positive integer with $j \leq N$ then all of the prime factors p of j satisfy $p \leq N$. Since $\{2,3,\dots,p_k\}$ is the set of all primes less than $N+1$, this means $j = 2^{m_1}\dots p_k^{m_k}$ for some integers $m_1,\dots,m_k$ with $0 \leq m_i$ for each i. From number theory, the choice of $m_1,\dots,m_k$ is unique (see [2, Ch.2]). This means there exists $R_N \subset \{j \in \boldsymbol{Z} | j > N\}$ such that

$$\{2^{m_1}\dots p_k^{m_k} | m_1,\dots,m_k \in \boldsymbol{N}\cup\{0\}\} = \{n \in \boldsymbol{Z} | n \leq N\} \cup R_N.$$

Thus,

$$\prod_{p\leq N}(1-f(p)p^{-s})^{-1} = \sum_{n=1}^{N} f(n)n^{-s} + \sum_{n\in R_N} f(n)n^{-s}.$$

Since $R_N \subseteq \{j \in \boldsymbol{Z} | j > N\}$, we have $\sum_{n\in R_N}|f(n)n^{-s}| \leq \sum_{n>N}|f(n)n^{-s}|$. Now, because $\sum_{n=1}^{\infty}|f(n)n^{-s}|$ converges,

$$\begin{aligned}\lim_{N\to\infty}\sum_{n=1}^{N}|f(n)n^{-s}| &= \lim_{N\to\infty}\left(\sum_{n=1}^{\infty}|f(n)n^{-s}| - \sum_{n>N}|f(n)n^{-s}|\right)\\ &= \sum_{n=1}^{\infty}|f(n)n^{-s}| - \lim_{N\to\infty}\sum_{n>N}|f(n)n^{-s}|,\end{aligned}$$

which implies

$$0 = \lim_{N\to\infty}\sum_{n>N}|f(n)n^{-s}| \geq \lim_{N\to\infty}\sum_{n\in R_N}|f(n)n^{-s}| \geq 0,$$

and so $\lim_{N\to\infty}\sum_{n\in R_N} f(n)n^{-s} = 0$ as well. This gives

$$\begin{aligned}\prod_{p}(1-f(p)p^{-s})^{-1} &= \lim_{N\to\infty}\prod_{p\leq N}(1-f(p)p^{-s})^{-1}\\ &= \sum_{n=1}^{\infty} f(n)n^{-s}. \ \square\end{aligned}$$

Consider the trivial Dirichlet character modulo 1, I_0. Since every positive integer is relatively prime to 1, we have $I_0(n) = 1$ for all positive integers n. Note that by Proposition 1.1, $\sum_{n=1}^{\infty} I_0(n)n^{-s} = \sum_{n=1}^{\infty} n^{-s}$ converges absolutely whenever $\Re(s) > 1$, converges uniformly on $\{s \in \boldsymbol{C} \mid \Re(s) > 1 + \delta\}$ whenever $\delta > 0$, and that $L_{I_0}(s) = \sum_{n=1}^{\infty} n^{-s}$ is a Dirichlet L-function. By Theorem 3, the Euler product representation for $\zeta(s)$ is $\prod_p (1 - I_0(p)p^{-s})^{-1} = \prod_p (1 - p^{-s})^{-1}$ whenever $\Re(s) > 1$.

Definition 2.1. For $s \in \boldsymbol{C}$, $\Re(s) > 1$, define the Riemann zeta funtion by $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

As noted above, ζ can be interpreted as the Dirichlet L-function associated with the trivial Dirichlet character modulo 1.

Proposition 2.2. Consider ζ as a function of a real variable s. Then

$$\lim_{s \to 1^+} (s-1)\zeta(s) = 1.$$

Proof: Let $s \in \boldsymbol{R}$, $s > 1$. We know t^{-s} is a monotone decreasing function of t when $t > 0$. This means for $n \geq 1$, $n \in \boldsymbol{Z}$,

$$(n+1)^{-s} < \int_n^{n+1} t^{-s} dt < n^{-s}.$$

Now, because $s > 1$,

$$\int_1^{\infty} t^{-s} dt = \left[\frac{t^{-s+1}}{1-s}\right]_1^{\infty} = \frac{1}{s-1},$$

so in fact,

$$\zeta(s) - 1 = \sum_{n=1}^{\infty} (n+1)^{-s} < \sum_{n=1}^{\infty} \int_n^{n+1} t^{-s} dt = \int_1^{\infty} t^{-s} dt = \frac{1}{s-1} < \sum_{n=1}^{\infty} n^{-s} = \zeta(s).$$

This gives

$$\zeta(s) - 1 < \frac{1}{s-1} < \zeta(s)$$
$$\implies -1 < \frac{1}{s-1} - \zeta(s) < 0$$

$$\begin{aligned}
&\implies 1+\frac{1}{s-1} > \zeta(s) > \frac{1}{s-1}\\
&\implies s > (s-1)\zeta(s) > 1\\
&\implies 1 = \lim_{s\to 1^+} s \geq \lim_{s\to 1^+}(s-1)\zeta(s) \geq 1.
\end{aligned}$$

Therefore $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$, as we wanted. $\square$

Corollary. Consider ζ as a function of a real variable s. Then

$$\lim_{s\to 1^+} \frac{\log \zeta(s)}{\log\left[(s-1)^{-1}\right]} = 1.$$

Proof: For $s > 1$ we have

$$\begin{aligned}
\log \zeta(s) &= \log\left[(s-1)^{-1}(s-1)\zeta(s)\right]\\
&= \log\left[(s-1)^{-1}\right] + \log\left[(s-1)\zeta(s)\right],
\end{aligned}$$

from which it follows that

$$\frac{\log \zeta(s)}{\log\left[(s-1)^{-1}\right]} = 1 + \frac{\log\left[(s-1)\zeta(s)\right]}{\log\left[(s-1)^{-1}\right]},$$

and so

$$\lim_{s\to 1^+} \frac{\log \zeta(s)}{\log\left[(s-1)^{-1}\right]} = \lim_{s\to 1^+}\left(1 + \frac{\log\left[(s-1)\zeta(s)\right]}{\log\left[(s-1)^{-1}\right]}\right).$$

Now, by Proposition 2.2, $\lim_{s\to 1^+}(s-1)\zeta(s) = 1$, so $\lim_{s\to 1^+}\log\left[(s-1)\zeta(s)\right] = 0$ and we get

$$\lim_{s\to 1^+} \frac{\zeta(s)}{\log\left[(s-1)^{-1}\right]} = 1. \ \square$$

Proposition 2.3. $\sum_p \sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks}$ converges absolutely whenever $\Re(s) > 1$, and converges uniformly on $\{s \in \boldsymbol{C} \mid \Re(s) > 1+\delta\}$ whenever $\delta > 0$, where the outer sum is over all primes p. Furthermore, $\exp\left(\sum_p \sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks}\right) = L_\chi(s)$ whenever $\Re(s) > 1$.

Proof: Consider $\sum_p \sum_{k=1}^{\infty} p^{-ks} = \sum_p \sum_{k=1}^{\infty}\left(p^k\right)^{-s}$. Define $\eta : \boldsymbol{N} \to \boldsymbol{Z}$ by $\eta(n) = 1$ if $n = p^k$ for some prime p and positive integer k, and $\eta(n) = 0$ otherwise. By Proposition 1.1, since $|\eta(n)| \leq 1$ for all n, $\sum_{n=1}^{\infty}\eta(n)n^{-s}$ converges absolutely whenever

$\Re(s) > 1$ and uniformly on $\{s \in \boldsymbol{C} \mid \Re(s) > 1 + \delta\}$ whenever $\delta > 0$. But $\sum_{n=1}^{\infty} \eta(n) n^{-s}$ is precisely $\sum_p \sum_{k=1}^{\infty} p^{-ks}$, so the latter converges absolutely and uniformly under the same conditions. Since $|(1/k)\chi(p)^k p^{-ks}| \leq p^{-ks}$ for all primes p and positive integers k, $\sum_p \sum_{k=1}^{\infty} (1/k)\chi(p)^k p^{-ks}$ is also absolutely and unformly convergent under the same conditions, as we wanted.

Next, recall that $\mathrm{Log}(1-x)$ has the power series representation $-\sum_{k=1}^{\infty} x^k/k$ absolutely convergent for $|x| < 1$ and uniformly convergent on $\{x \in \boldsymbol{C} \mid |x| < 1 - \delta\}$ whenever $\delta > 0$. Since $|\chi(p)p^{-s}| \leq |p^{-s}| < 1$ whenever $\Re(s) > 1$, the series $\sum_{k=1}^{\infty}(1/k)\chi(p^k)p^{-ks}$ converges to $-\mathrm{Log}\left((1-\chi(p)p^{-s})\right)$ whenever $\Re(s) > 1$, for all primes p. Thus,

$$\exp\left(\sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks}\right) = (1-\chi(p)p^{-s})^{-1}.$$

Let N be some positive integer. If p_i denotes the ith prime and p_t is the largest prime less than or equal to N, then

$$\sum_{p \leq N} \sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks} = -\mathrm{Log}\left((1-\chi(2)2^{-ks})\right) - \cdots - \mathrm{Log}\left((1-\chi(p_t)p_t^{-ks})\right),$$

which implies

$$\exp\left(\sum_{p \leq N} \sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks}\right) = \prod_{p \leq N}(1-\chi(p)p^{-s})^{-1}. \tag{1}$$

By Theorem 3,

$$L_\chi(s) \quad = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1} = \lim_{N \to \infty} \prod_{p \leq N}(1-\chi(p)p^{-s})^{-1}.$$

Since the exponential function is continuous, taking the limit as $N \to \infty$ in (1) gives precisely $\exp\left(\sum_p \sum_{k=1}^{\infty}(1/k)\chi(p)^k p^{-ks}\right) = L_\chi(s)$, as we wanted. $\square$

Corollary. Assume $\Re(s) > 1$. Then $\zeta(s) \neq 0$.

Proof: By Proposition 2.3, $\sum_p \sum_{k=1}^{\infty}(1/k)p^{-ks}$ converges absolutely, and

$$\exp\left(\sum_p \sum_{k=1}^{\infty}(1/k)p^{-ks}\right) = \zeta(s).$$

This implies that $\sum_p \sum_{k=1}^{\infty}(1/k)p^{-ks} = \mathrm{Log}(\zeta(s)) + 2\pi i q$ for some integer q. In particular, $\mathrm{Log}(\zeta(s))$ is defined, from which it follows that $\zeta(s) \neq 0$. $\square$

Definition 2.2. Define $G_\chi : \{s \in \boldsymbol{R} | s > 1\} \to \boldsymbol{C}$ by $G_\chi(s) = \sum_p \sum_{k=1}^{\infty} (1/k)\chi(p)^k p^{-ks}$ for every $s > 1$ (where the outer sum is over all primes p). By Proposition 2.3, the series for G_χ converges absolutely, and is uniformly convergent on $\{s \in \boldsymbol{R} \mid s > 1 + \delta\}$ whenever $\delta > 0$. We refer to G as the logarithm associated with L_χ.

Note that $G_\chi(s)$ is not *a priori* equal to $\mathrm{Log}(L_\chi(s))$. Instead, $G_\chi(s)$ is the sum over all primes p of $-\mathrm{Log}(1 - \chi(p)p^{-s})$. Each of these logarithms may have a nonzero imaginary part, and the sum over all primes of these imaginary parts gives $\Im(G_\chi(s))$. By definition, $|\Im(\mathrm{Log}(z))| \leq \pi$ for all $z \in \boldsymbol{C}\backslash\{0\}$. On the other hand, there is no guarantee that $|\Im(G_\chi(s))| \leq \pi$. We work with the branch of the logarithm specified by G_χ in order to have an unambiguous, easily manipulable expression for the logarithm of L_χ.

Proposition 2.4 G_χ is continuous on $(1, \infty)$.

Proof: We have $G_\chi(s) = \sum_p \sum_{k=1}^{\infty} (1/k)\chi(p)^k p^{-ks}$ uniformly convergent on $\{s \in \boldsymbol{R} \mid s > 1 + \delta\}$ whenever $\delta > 0$. For each positive integer k and prime p,

$$s \mapsto (1/k)\chi(p)^k p^{-ks}$$

is clearly continuous for $s > 1$. Uniform convergence then ensures G_χ is continuous on $(1, \infty)$ as well (see [3, Ch.20]). $\square$

We now express G_χ in a new way, which will be necessary for the proof of Dirichlet's theorem. We separate the series for G_χ into a sum of two terms, one of which is bounded as $s \to 1^+$, and one of which will be investigated further in the next section. We will find eventually that in fact G_χ is bounded as $s \to 1^+$ whenever χ is nontrivial. This will be the most difficult result necessary for our proof of Dirichlet's theorem.

Proposition 2.5. Assume $s > 1$. Then

$$G_\chi(s) = \sum_{p \nmid m} \chi(p)p^{-s} + R_\chi(s)$$

for some function R_χ, where R_χ remains bounded as $s \to 1^+$.

Proof: By definition, $G_\chi(s) = \sum_p \sum_{k=1}^{\infty} (1/k)\chi(p)^k p^{-ks}$. By Proposition 2.3 this series is absolutely convergent for $s > 1$, so we can separate the terms with $k = 1$, and note that $\chi(p) = 0$ if $p|m$, to get

$$G_\chi(s) = \sum_{p\nmid m} \chi(p)p^{-s} + \sum_p \sum_{k=2}^{\infty} (1/k)\chi(p)^k p^{-ks}.$$

Write $R_\chi(s) = \sum_p \sum_{k=2}^{\infty} (1/k)\chi(p)^k p^{-ks}$, so we have $G_\chi(s) = \sum_{p\nmid m} \chi(p)p^{-s} + R_\chi(s)$. Here, $|(1/k)\chi(p)^k p^{-ks}| \leq p^{-ks}$ for all positive integers k, so

$$|R_\chi(s)| \quad \leq \sum_p \sum_{k=2}^{\infty} |(1/k)\chi(p)^k p^{-ks}| \leq \sum_p \sum_{k=2}^{\infty} p^{-ks}.$$

Now, $p^{-s} < 1$ for p prime, so using the usual expression for geometric series we have

$$\sum_p \sum_{k=2}^{\infty} p^{-ks} \quad = \sum_p p^{-2s}(1 - p^{-s})^{-1} \leq (1 - 2^{-s})^{-1} \sum_p p^{-2s},$$

since clearly $(1 - p^{-s})^{-1} \leq (1 - 2^{-s})^{-1}$ for all primes p, $s > 1$. Now, for $s > 1$ we have $p^{-2s} \leq p^{-2}$ and $(1 - 2^{-s})^{-1} \leq (1 - 1/2)^{-1} = 2$, so in fact,

$$\sum_p \sum_{k=2}^{\infty} p^{-ks} \quad \leq (1 - 2^{-s})^{-1} \sum_p p^{-2s} \leq 2 \sum_p p^{-2} = 2\zeta(2).$$

This implies $|R_\chi(s)| \leq 2\zeta(2)$ as well. Here $\sum_{p\nmid m} \chi(p)p^{-s}$ is clearly uniformly convergent on $\{s \in \boldsymbol{R} \mid s > 1 + \delta\}$ whenever $\delta > 0$ (by Proposition 1.1), and $\chi(p)p^{-s}$ is clearly continuous. Thus both $\sum_{p\nmid m} \chi(p)p^{-s}$ and $G_\chi(s)$ are continuous, and so $R_\chi(s)$ is as well. This gives $\lim_{s\to 1^+} |R_\chi(s)| \leq 2\zeta(2)$, so $R_\chi(s)$ is bounded as $s \to 1^+$, as we wanted. $\square$

IV. DIRICHLET'S THEOREM

We are now in a position to begin the proof of Dirichlet's theorem. In this section we will state the theorem and prove some preliminary results. The most difficult part of the proof will be reserved for its own section to follow. Again let m be a positive integer throughout, and let χ_0 denote the trivial Dirichlet character modulo m.

Definition 3.1. Let $\mathcal{P}$ be a set of primes. Define the Dirichlet density of $\mathcal{P}$ by

$$d(\mathcal{P}) = \lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}} p^{-s}}{\log\left[(s-1)^{-1}\right]},$$

if the limit exists.

Lemma 3.1. Let $\mathcal{P}$ be a set of primes.

(a) If $\mathcal{P}$ is finite then $d(\mathcal{P}) = 0$.

(b) If $\mathcal{P}$ consists of all but finitely many primes then $d(\mathcal{P}) = 1$.

(c) If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint and $d(\mathcal{P}_1)$, $d(\mathcal{P}_2)$ exist, then $d(\mathcal{P}) = d(\mathcal{P}_1) + d(\mathcal{P}_2)$.

Proof: To prove (a), note that if $\mathcal{P}$ is finite then $\sum_{p\in\mathcal{P}} p^{-s}$ is bounded as $s \to 1^+$. Since $\log[(s-1)^{-1}]$ is clearly unbounded as $s \to 1^+$, we are done.

To prove (b) let I_0 be the trivial Dirichlet character modulo 1, and note that by Proposition 2.3, for real $s > 1$ we have $\exp(G_{I_0}(s)) = L_{I_0}(s) = \zeta(s)$. Since

$$G_{I_0}(s) = \sum_p \sum_{k=1}^{\infty} (1/k) I_0(p)^k p^{-ks} = \sum_p \sum_{k=1}^{\infty} (1/k) p^{-ks}$$

is real for real $s > 1$, we must have $\log \zeta(s) = G_{I_0}(s)$ (recall that log denotes the real logarithm). If $\mathcal{P}$ consists of all but finitely many primes, we can write

$$\sum_{p\in\mathcal{P}} p^{-s} = \sum_p p^{-s} - \sum_{p\in\mathcal{F}} p^{-s},$$

where $\mathcal{F}$ is some finite set of primes. Using part (a), this gives

$$\begin{aligned} d(\mathcal{P}) &= \lim_{s\to 1^+} \left(\frac{\sum_p p^{-s} - \sum_{p\in\mathcal{F}} p^{-s}}{\log[(s-1)^{-1}]} \right) \\ &= \lim_{s\to 1^+} \frac{\sum_p p^{-s}}{\log[(s-1)^{-1}]}, \end{aligned}$$

if the limit exists. By Proposition 2.5,

$$\log \zeta(s) = G_{I_0}(s) = \sum_p I_0(p) p^{-s} + R(s) = \sum_p p^{-s} + R(s)$$

where $R(s)$ remains bounded as $s \to 1^+$. Thus again by part (a),

$$\lim_{s\to 1^+} \frac{\log \zeta(s)}{\log[(s-1)^{-1}]} = \lim_{s\to 1^+} \frac{\sum_p p^{-s}}{\log[(s-1)^{-1}]},$$

if the limit exists. In other words,

$$d(\mathcal{P}) = \lim_{s\to 1^+} \frac{\log \zeta(s)}{\log[(s-1)^{-1}]}$$

as long as the limit on the right exists. The corollary to to Proposition 2.2 tells us that the limit on the right is equal to 1, and so $d(\mathcal{P}) = 1$ as we wanted.

Finally, to prove (c), note that for $s > 1$ we have

$$\sum_{p \in \mathcal{P}} p^{-s} = \sum_{p \in \mathcal{P}_1} p^{-s} + \sum_{p \in \mathcal{P}_2} p^{-s} \tag{2}$$

(all of the series in (2) are bounded by $\zeta(s)$ and hence converge absolutely). By assumption,

$$d(\mathcal{P}_1) = \lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}_1} p^{-s}}{\log\left[(s-1)^{-1}\right]}$$

and

$$d(\mathcal{P}_2) = \lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}_2} p^{-s}}{\log\left[(s-1)^{-1}\right]}$$

both exist, so

$$\begin{aligned} d(\mathcal{P}) &= \lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}} p^{-s}}{\log\left[(s-1)^{-1}\right]} \\ &= \lim_{s \to 1^+} \frac{\sum_{p \in \mathcal{P}_1} p^{-s} + \sum_{p \in \mathcal{P}_2} p^{-s}}{\log\left[(s-1)^{-1}\right]} \end{aligned}$$

exists as well, and is equal to $d(\mathcal{P}_1) + d(\mathcal{P}_2)$ as we wanted. □

We now state Dirichlet's theorem. Several more results will be needed before a proof of the theorem can be given.

Theorem 4 [Dirichlet's Theorem]. Suppose a and m are relatively prime integers, with $m > 1$. Let $\mathcal{P}(a; m)$ be the set of primes p with $p \equiv a \pmod{m}$. Then $d(\mathcal{P}(a; m)) = 1/\phi(m)$. In particular, $\mathcal{P}(a; m)$ is infinite.

We will now show that for $\chi \in \Delta_m$, L_χ can be analytically continued to the region $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$ if χ is nontrivial and to $\{s \in \boldsymbol{C} \mid \Re(s) > 0,\ s \neq 1\}$ if $\chi = \chi_0$. The continuations can be constructed by representing the L-functions in integral form. We will first need a lemma regarding representations of series.

Lemma 3.2. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of complex numbers, and suppose $\sum_{n=1}^{\infty} a_n b_n$ converges. Further suppose $(a_1 + a_2 + \ldots + a_n)b_n \to 0$ as $n \to \infty$.

Then

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} (a_1 + ... + a_n)(b_n - b_{n+1}).$$

Proof: Let $A_0 = 0$, $A_n = a_1 + ... + a_n$ for each $n \geq 1$. Then $A_n - A_{n-1} = a_n$, so for each positive integer N,

$$\sum_{n=1}^{N} a_n b_n \quad = \sum_{n=1}^{N} (A_n - A_{n-1}) b_n = \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n.$$

Since $A_0 = 0$,

$$\sum_{n=1}^{N} A_{n-1} b_n = \sum_{n=2}^{N} A_{n-1} b_n = \sum_{n=1}^{N-1} A_n b_{n+1},$$

and we get

$$\begin{aligned} \sum_{n=1}^{N} a_n b_n \quad &= \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N} A_{n-1} b_n \\ &= \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} \\ &= A_N b_N + \sum_{n=1}^{N-1} A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} \\ &= A_N b_N + \sum_{n=1}^{N-1} A_n (b_n - b_{n+1}), \end{aligned}$$

which implies

$$\sum_{n=1}^{\infty} a_n b_n \quad = \lim_{N \to \infty} \sum_{n=1}^{N} a_n b_n = \lim_{N \to \infty} \left(A_N b_N + \sum_{n=1}^{N} A_n (b_n - b_{n+1}) \right).$$

Here $A_N b_N = (a_1 + ... + a_N) b_N$, so by our hypotheses $A_N b_N \to 0$ as $N \to \infty$. Thus

$$\sum_{n=1}^{\infty} a_n b_n \quad = \lim_{N \to \infty} \left(A_N b_N + \sum_{n=1}^{N} A_n (b_n - b_{n+1}) \right) = \sum_{n=1}^{\infty} A_n (b_n - b_{n+1}),$$

or $\sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} (a_1 + ... + a_n)(b_n - b_{n+1})$, as we wanted. □

Lemma 3.3. Let $s \in \boldsymbol{C}$, $\Re(s) > 1$. Then

$$L_{\chi_0}(s) = \left(\prod_{p|m} (1 - p^{-s}) \right) \zeta(s).$$

Proof: By definition, $L_{\chi_0}(s) = \sum_{n=1}^{\infty} \chi_0(n) p^{-s}$. Since χ_0 is multiplicative (Lemma 1.3) and $|\chi_0(n)| \leq 1$ for all positive integers n, Theorem 3 gives $L_{\chi_0}(s) = \prod_p (1 - \chi_0(p)p^{-s})^{-1}$, where the product is over all primes p. As χ_0 is the trivial Dirichlet character modulo m, for an integer n we have $\chi_0(n) = 0$ if $(n, m) \neq 1$ and $\chi_0(n) = 1$ otherwise. In particular, for p prime, $(1 - \chi_0(p)p^{-s})^{-1} = 1$ if $p|m$ and $(1 - \chi_0(p)p^{-s})^{-1} = (1 - p^{-s})^{-1}$ if $p \nmid m$. Thus

$$L_{\chi_0}(s) = \prod_p (1 - \chi_0(p)p^{-s})^{-1} = \prod_{p \nmid m} (1 - p^{-s})^{-1}.$$

Now recall that by definition $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Theorem 3 again gives a product decomposition, $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$. Therefore, we get

$$\begin{aligned} L_{\chi_0}(s) &= \prod_{p \nmid m} (1 - p^{-s})^{-1} \\ &= \left(\prod_{p|m} (1 - p^{-s}) \right) \left(\prod_p (1 - p^{-s})^{-1} \right) \\ &= \left(\prod_{p|m} (1 - p^{-s}) \right) \zeta(s). \ \square \end{aligned}$$

Proposition 3.1. $\zeta(s) - (s - 1)^{-1}$ can be analytically continued to the domain $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$.

Proof: Assume $\Re(s) > 1$. Then, by definition, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, where the series is absolutely convergent (Proposition 1.1). Write $a_n = 1$ and $b_n = n^{-s}$, for each positive integer n. Then for each positive integer n, $a_1 + a_2 + ... + a_n = n$. Since $\Re(s) > 1$, $(a_1 + ... + a_n)b_n = nn^{-s} = n^{-s+1} \to 0$ as $n \to \infty$. Thus, by Lemma 3.2

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} n^{-s} \\ &= \sum_{n=1}^{\infty} a_n b_n \\ &= \sum_{n=1}^{\infty} (a_1 + ... + a_n)(b_n - b_{n+1}) \\ &= \sum_{n=1}^{\infty} n(n^{-s} - (n + 1)^{-s}). \end{aligned}$$

Now, note that

$$n^{-s} - (n+1)^{-s} = s \int_n^{n+1} x^{-s-1} dx,$$

so that

$$\zeta(s) = \sum_{n=1}^{\infty} n(n^{-s} - (n+1)^{-s}) = s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1} dx.$$

For a real number x, let $[x]$ denote the largest integer smaller than or equal to x, and write the fractional part of x as $\langle x \rangle = x - [x]$. Obviously, $0 \leq \langle x \rangle < 1$ for all real x. We can rewrite our new expression for $\zeta(s)$ using this notation by noting that if $n \leq x \leq n+1$ where n is an integer, then $n = [x]$. We get

$$\begin{aligned}
\zeta(s) &= s \sum_{n=1}^{\infty} n \int_n^{n+1} x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \int_n^{n+1} n x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \int_n^{n+1} [x] x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \int_n^{n+1} (x - \langle x \rangle) x^{-s-1} dx \\
&= s \sum_{n=1}^{\infty} \int_n^{n+1} \left(x^{-s} - \langle x \rangle x^{-s-1} \right) dx. \qquad (3)
\end{aligned}$$

Now, since $\Re(s) > 1$,

$$\int_1^{\infty} x^{-s} dx = \left[\frac{x^{-s+1}}{1-s} \right]_1^{\infty} = \frac{1}{s-1}$$

and for $\Re(s) > 0$, since $|\langle x \rangle| < 1$,

$$\begin{aligned}
\left| \int_1^{\infty} \langle x \rangle x^{-s-1} dx \right| &\leq \int_1^{\infty} |\langle x \rangle x^{-s-1}| dx \\
&< \int_1^{\infty} |x^{-s-1}| dx \\
&= \int_1^{\infty} x^{-\Re(s)-1} dx \\
&= \left[-\frac{x^{-\Re(s)}}{\Re(s)} \right]_1^{\infty} \\
&= \frac{1}{\Re(s)}.
\end{aligned}$$

Here we have used the fact that for real x and complex s, $|x^s| = x^{\Re(s)}$, as demonstrated in Section I. In particular, this means that $\int_1^\infty (x^{-s} - \langle x \rangle x^{-s-1})\, dx$ converges. Returning to (3), we proceed to see

$$\begin{aligned}\zeta(s) &= s\sum_{n=1}^{\infty}\int_n^{n+1}\left(x^{-s} - \langle x\rangle x^{-s-1}\right)dx \\ &= s\int_1^\infty \left(x^{-s} - \langle x\rangle x^{-s-1}\right)dx \\ &= s\int_1^\infty x^{-s}dx - s\int_1^\infty \langle x\rangle x^{-s-1}dx \\ &= \frac{s}{s-1} - s\int_1^\infty \langle x\rangle x^{-s-1}dx,\end{aligned}$$

or

$$\begin{aligned}\zeta(s) - (s-1)^{-1} &= \frac{s}{s-1} - \frac{1}{s-1} - s\int_1^\infty \langle x\rangle x^{-s-1}dx \\ &= 1 - s\int_1^\infty \langle x\rangle x^{-s-1}dx.\end{aligned}$$

Here, $\langle x\rangle x^{-s-1}$ is clearly analytic for all $s \in \boldsymbol{C}$ whenever $x \geq 1$, and we have already shown $\int_1^\infty \langle x\rangle x^{-s-1}dx$ converges when $\Re(s) > 0$, so in fact this integral defines an analytic function of s for $\Re(s) > 0$ (see [5]). Moreover, $Z(s) = 1 - \int_1^\infty \langle x\rangle x^{-s-1}dx$ is the analytic continuation of $\zeta(s) - (s-1)^{-1}$ to the domain $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$, as we wanted. □

Corollary. L_{χ_0} can be analytically continued to the domain $\{s \in \boldsymbol{C} \mid \Re(s) > 0, s \neq 1\}$. Furthermore, the analytic continuation has a simple pole at $s = 1$.

Proof: Let $Z(s)$ be the analytic continuation of $\zeta(s) - (s-1)^{-1}$ to $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$ as in the proof of Proposition 3.1, and let $C = \prod_{p|m}(1 - p^{-s})$. Then by Lemma 3.3, when $\Re(s) > 1$ we have

$$L_{\chi_0}(s) \quad = C\zeta(s) = C(Z(s) + (s-1)^{-1}).$$

Clearly $(s-1)^{-1}$ is analytic whenever $s \neq 1$, and by definition, $Z(s)$ is as well. Thus $\mathcal{L}_{\chi_0}(s) = C(Z(s) + (s-1)^{-1})$ is the analytic continuation of L_{χ_0} to $\{s \in \boldsymbol{C} \mid \Re(s) > 0, s \neq 1\}$, as we wanted.

Since $Z(s)$ is analytic at $s = 1$ and $(s-1)^{-1}$ has a simple pole at $s = 1$, it is clear that $\mathcal{L}_{\chi_0}(s)$ has a simple pole at $s = 1$. □

Lemma 3.4. Let χ be a nontrivial Dirichlet character modulo m. Then $|\sum_{n=0}^{N}\chi(n)| \leq \phi(m)$ for all positive integers N.

Proof: By the division algorithm (see [2, Ch.2]), we can express N in the form $N = qm+r$, where q and r are integers and $0 \leq r < m$. Write

$$\sum_{n=1}^{N}\chi(n) = \sum_{n=0}^{m-1}\chi(n) + \sum_{n=0}^{m-1}\chi(n+m) + ... + \sum_{n=0}^{m-1}\chi(n+(q-1)m) + \sum_{n=0}^{r}\chi(n+qm).$$

By Lemma 1.4, $\chi(n+km) = \chi(n)$ for all positive integers n and k. This gives

$$\begin{aligned}\sum_{n=1}^{N}\chi(n) &= \sum_{n=0}^{m-1}\chi(n) + ... + \sum_{n=0}^{m-1}\chi(n+(q-1)m) + \sum_{n=0}^{r}\chi(n+qm)\\ &= q\sum_{n=0}^{m-1}\chi(n) + \sum_{n=0}^{r}\chi(n).\end{aligned}$$

Note that whenever $\chi(n) \neq 0$ we have $\chi_0(n) = 1$, so $\sum_{n=0}^{m-1}\chi(n) = \sum_{n=0}^{m-1}\overline{\chi_0(n)}\chi(n)$. Since by assumption $\chi \neq \chi_0$, part (a) of Corollary 1.2.1 gives $\sum_{n=0}^{m-1}\overline{\chi_0(n)}\chi(n) = 0$, and so $\sum_{n=0}^{m-1}\chi(n) = 0$ as well. Thus

$$\sum_{n=1}^{N}\chi(n) = q\sum_{n=0}^{m-1}\chi(n) + \sum_{n=0}^{r}\chi(n) = \sum_{n=0}^{r}\chi(n),$$

and, since $r \leq m-1$,

$$\Big|\sum_{n=1}^{N}\chi(n)\Big| = \Big|\sum_{n=0}^{r}\chi(n)\Big| \leq \sum_{n=0}^{r}|\chi(n)| \leq \sum_{n=0}^{m-1}|\chi(n)|.$$

By definition, $|\chi(n)| = 1$ when $n+m\boldsymbol{Z} \in U(\boldsymbol{Z}_m)$ and $\chi(n) = 0$ otherwise. Also by definition $|U(\boldsymbol{Z}_m)| = \phi(m)$, and clearly $n_1 + m\boldsymbol{Z} \neq n_2 + m\boldsymbol{Z}$ when $n_1 \neq n_2$ and $0 \leq n_1, n_2 < m$. Together these give $\phi(m) \geq \sum_{n=0}^{m-1}|\chi(n)| \geq |\sum_{n=1}^{N}\chi(n)|$, as we wanted. □

Proposition 3.2. Let χ be a nontrivial Dirichlet character modulo m. Then L_χ can be analytically continued to the domain $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$.

Proof: Assume $\Re(s) > 1$. Then by definition,

$$L_\chi(s) = \sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

Let $a_n = \chi(n)$, $b_n = n^{-s}$ for each positive integer n. Then $|(a_1+...+a_n)| \leq |a_1|+...+|a_n| \leq n$ for every positive integer n. Since $\Re(s) > 1$, this means

$$|(a_1+...+a_n)b_n| \leq n|b_n| = n|n^{-s}| = nn^{-\Re(s)} = n^{-\Re(s)+1}.$$

Since $n^{-Re(s)+1}$ decreases to 0 as n increases to infinity, $|(a_1 + ... + b_n)b_n|$ does as well. Furthermore, since $L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \sum_{n=1}^{\infty} a_n b_n$ converges, by Lemma 3.2 we have

$$\begin{aligned} L_\chi(s) &= \sum_{n=1}^{\infty} a_n b_n \\ &= \sum_{n=1}^{\infty} (a_1 + ... + a_n)(b_n - b_{n+1}) \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \chi(k)(n^{-s} - (n+1)^{-s}) \right). \end{aligned}$$

As in the proof of Proposition 3.1, write

$$n^{-s} - (n+1)^{-s} = s \int_n^{n+1} x^{-s-1} dx,$$

so that

$$L_\chi(s) = s \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \chi(k) \right) \int_n^{n+1} x^{-s-1} dx.$$

Define $S(x) = \sum_{n \le x} \chi(n)$ for all positive real numbers x, where the sum is over all positive integers n less than or equal to x. Then for $n \le x < n+1$, we have $S(x) = S(n)$, so

$$\begin{aligned} L_\chi(s) &= s \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \chi(k) \right) \int_n^{n+1} x^{-s-1} dx \\ &= s \sum_{n=1}^{\infty} S(n) \int_n^{n+1} x^{-s-1} dx \\ &= s \sum_{n=1}^{\infty} \int_n^{n+1} S(x) x^{-s-1} dx. \end{aligned}$$

By Lemma 3.4, we have $|S(x)| \le \phi(m)$ for all $x \ge 1$, so, for $\Re(s) > 0$,

$$\begin{aligned} \left| \int_1^{\infty} S(x) x^{-s-1} dx \right| &\le \int_1^{\infty} |S(x) x^{-s-1}| dx \\ &\le \int_1^{\infty} \phi(m) x^{-\Re(s)-1} dx \\ &= \phi(m) \left[\frac{-x^{-\Re(s)}}{\Re(s)} \right]_1^{\infty} \\ &= \frac{\phi(m)}{\Re(s)}. \end{aligned}$$

In particular, we see that $\int_1^{\infty} S(x) x^{-s-1} dx$ converges, and thus

$$L_\chi(s) = s \sum_{n=1}^{\infty} \int_n^{n+1} S(x) x^{-s-1} dx = s \int_1^{\infty} S(x) x^{-s-1} dx.$$

Clearly $S(x)x^{-s-1}$ is analytic whenever $\Re(s) > 0$, and we have already shown that the integral on the right converges whenever $\Re(s) > 0$. Thus the integral defines an analytic function on $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$ (see [5]). Here $\mathcal{L}_\chi(s) = s\int_1^\infty S(x)x^{-s-1}dx$ is the analytic continuation of L_χ to the domain $\{s \in \boldsymbol{C} \mid \Re(s) > 0\}$, as we wanted. $\square$

Henceforth, whenever we refer to L_χ we actually refer to the analytic continuation constructed above, unless otherwise specified.

Corollary. If $\chi \in \Delta_m$ is nontrivial, then $L_\chi(s)$ is bounded as $s \to 1$.

V. THE BOUNDEDNESS OF G_χ

Once again let χ_0 denote the trivial Dirichlet character modulo a positive integer m, and let G_χ denote the logarithm associated with L_χ for a Dirichlet character χ modulo m, as in Definition 2.2. In order to show that $G_\chi(s)$ is bounded as $s \to 1^+$, we will require that $L_\chi(1) \neq 0$. Otherwise, since G_χ is a branch of the logarithm of L_χ, it will diverge as $s \to 1^+$.

Lemma 4.1. Let $s > 1$ be real. Then $\prod_{\chi\in\Delta_m} L_\chi(s)$ is real, and $\prod_{\chi\in\Delta_m} L_\chi(s) > 1$.

Proof: Let $\chi \in \Delta_m$. Then by definition

$$G_\chi(s) = \sum_p \sum_{k=1}^{\infty} (1/k)\chi(p^k)p^{-ks},$$

and the series is absolutely convergent for all $\chi \in \Delta_m$ (Proposition 2.3). Thus

$$\begin{aligned}\sum_{\chi\in\Delta_m} G_\chi(s) &= \sum_{\chi\in\Delta_m}\sum_p \sum_{k=1}^{\infty} (1/k)\chi(p^k)p^{-ks} \\ &= \sum_p \sum_{k=1}^{\infty}\left((1/k)p^{-ks}\sum_{\chi\in\Delta_m}\chi(p^k)\right).\end{aligned}$$

Since $\chi(1) = 1$ we have

$$\sum_{\chi\in\Delta_m}\chi(p^k) = \sum_{\chi\in\Delta_m}\chi(p^k)\overline{\chi(1)}$$

for all primes p and positive integers k. If $p^k \equiv 1(\text{mod } m)$ then $(p,m)=1$ so $(p^k,m)=1$. By Corollary 1.2.1, the sum is then equal to $\phi(m)$. On the other hand, if $p^k \not\equiv 1 \pmod m$ then we may have either $(p^k,m)=1$ or $(p^k,m)\neq 1$. If $(p^k,m)=1$ then by Corollary 1.2.1, the sum is 0. If $(p^k,m)\neq 1$ then by definition $\chi(p^k)=0$ so the sum is again 0. As a result,

$$\sum_{\chi\in\Delta_m} G_\chi(s) \quad = \sum_p \sum_{k=1}^{\infty} \left((1/k)p^{-ks} \sum_{\chi\in\Delta_m} \chi(p^k) \right) = \phi(m) \sum_{p,k} (1/k)p^{-ks},$$

where the sum is now over all primes p and positive integers k such that $p^k \equiv 1(\text{mod } m)$. Here $\phi(m)(1/k)p^{-ks}$ is real and positive for all primes p and integers k, and clearly there exist a prime p and positive integer k such that $p^k \equiv 1 \pmod m$. Therefore, $\sum_{\chi(s)\in\Delta_m} G_\chi(s)$ is real and positive. In particular,

$$\exp\left(\sum_{\chi\in\Delta_m} G_\chi(s) \right) > 1.$$

By Proposition 2.3, for each $\chi \in \Delta_m$ we have $\exp(G_\chi(s)) = L_\chi(s)$. Thus

$$1 \quad < \exp\left(\sum_{\chi\in\Delta_m} G_\chi(s) \right) = \prod_{\chi\in\Delta_m} \exp(G_\chi(s)) = \prod_{\chi\in\Delta_m} L_\chi(s),$$

as we wanted. □

Lemma 4.2. Let $\delta > 0$, and let k be a nonnegative integer. Then

$$\lim_{n\to\infty} \frac{(\log n)^k}{n^\delta} = 0.$$

Proof: If $k=0$, the result is obvious. If $k=1$ then we are looking at

$$\lim_{n\to\infty} \frac{\log n}{n^\delta}.$$

Here we can use L'Hopital's rule (see [3, Ch.4]) to find

$$\lim_{n\to\infty} \frac{\log n}{n^\delta} \quad = \lim_{n\to\infty} \frac{\frac{1}{n}}{\delta n^{\delta-1}} = \lim_{n\to\infty} \frac{1}{\delta n^\delta} = 0.$$

Now suppose $k > 1$. Then

$$\lim_{n\to\infty} \frac{(\log n)^k}{n^\delta} = \left(\lim_{n\to\infty} \frac{\log n}{n^{\delta/k}} \right)^k,$$

if the limit on the right exists. By the argument above the limit on the right is precisely 0, and so we find

$$\lim_{n\to\infty} \frac{(\log n)^k}{n^\delta} = 0^k = 0. \ \square$$

Proposition 4.1. Let f be a nonnegative function defined on the positive integers and let $s > 1$. Assume that $f(n)f(m) = f(nm)$ whenever $(m,n) = 1$, and that there is a real number c such that $f(p^k) \leq c$ for every prime p and positive integer k. Then $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges and

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_p \left(1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks}\right),$$

where the product is over all primes p.

Proof: Here $|f(p^k)p^{-ks}| < cp^{-ks}$ for every prime p and positive integer k. Because $s > 1$, $p^{-s} < 1$ and $\sum_{k=1}^{\infty} p^{-ks}$ is an (absolutely) convergent geometric series. Thus, by comparison, $\sum_{k=1}^{\infty} f(p^k)p^{-ks}$ converges absolutely as well.

For convenience, let $a(p) = \sum_{k=1}^{\infty} f(p^k)p^{-ks}$ for each prime p. Since f is nonnegative by assumption, $a(p)$ is as well. We have $a(p) \leq c\sum_{k=1}^{\infty} p^{-ks}$. The series on the right is geometric and is equal to $p^{-s}(1-p^{-s})^{-1}$. For a prime p we have $(1-p^{-s})^{-1} < (1-2^{-1})^{-1} = 2$, so in fact, $a(p) < 2cp^{-s}$. Note $\sum_p p^{-s}$ converges since $s > 1$. Therefore we can let

$$M = 2c\sum_p p^{-s} \geq \sum_{p\leq N} a(p),$$

where the last inequality holds for all positive integers N.

Recall that the Taylor series for $\exp(x)$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + ...,$$

so in particular, for $x \geq 0$ we have $1 + x \leq \exp(x)$. Thus for each prime p, we have $(1 + a(p)) \leq \exp(a(p))$, and in particular,

$$\prod_{p\leq N}(1 + a(p)) \quad \leq \prod_{p\leq N} \exp(a(p)) = \exp\left(\sum_{p\leq N} a(p)\right).$$

Since exp is a monotone increasing function, this means that

$$\prod_{p\leq N}(1+a(p)) \quad \leq \exp\left(\sum_{p\leq N} a(p)\right) \leq \exp(M).$$

Recall $a(p) = \sum_{k=1}^{\infty} f(p^k)p^{-ks}$. Note that $(1,1) = 1$, so $f(1)^2 = f(1)f(1) = f(1)$, and thus $f(1) = 1$ since f is nonnegative by assumption. Then $1 + a(p) = \sum_{k=0}^{\infty} f(p^k)p^{-ks}$. Let N be any positive integer. We have

$$\begin{aligned}
\exp(M) \quad &\geq \prod_{p\leq N}(1+a(p)) \\
&= \prod_{p\leq N}\left(\sum_{k=0}^{\infty} f(p^k)p^{-ks}\right) \\
&= \left(\sum_{k_1=0}^{\infty} f(2^{k_1})2^{-k_1 s}\right)\left(\sum_{k_2=0}^{\infty} f(3^{k_2})3^{-k_2 s}\right)...\left(\sum_{k_r=0}^{\infty} f(p_r^{k_r})p_r^{-k_r s}\right),
\end{aligned}$$

where p_r denotes the largest prime smaller than or equal to N. Recall that each series on the right is absolutely convergent, so we can rearrange the terms to get

$$\exp(M) \quad \geq \prod_{p\leq N}(1+a(p)) = \sum_{k_1=0}^{\infty}...\sum_{k_r=0}^{\infty} f(2^{k_1})2^{-k_1 s}...f(p_r^{k_r})p_r^{-k_r s}.$$

Here, $2^{k_1}, ..., p_r^{k_r}$ are pairwise relatively prime for any positive integers $k_1, ..., k_r$. Thus by the assumptions on f,

$$f(2^{k_1})...f(p_r^{k_r}) = f(2^{k_1}...p_r^{k_r}).$$

As a result,

$$\begin{aligned}
\exp(M) \quad &\geq \prod_{p\leq N}(1+a(p)) \\
&= \sum_{k_1=0}^{\infty}...\sum_{k_r=0}^{\infty} f(2^{k_1})2^{-k_1 s}...f(p_r^{k_r})p_r^{-k_r s} \\
&= \sum_{k_1=0}^{\infty}...\sum_{k_r=0}^{\infty} f(2^{k_1}...p_r^{k_r})(2^{k_1}...p_r^{k_r})^{-s},
\end{aligned}$$

so by the same argument as in the proof of Theorem 3, there is some subset R_N of $\{j \in \boldsymbol{Z} | j > N\}$ with

$$\lim_{N\to\infty}\sum_{n\in R_N} f(n)n^{-s} = 0,$$

and

$$\exp(M) \geq \prod_{p \leq N} (1 + a(p)) = \sum_{n=1}^{N} f(n)n^{-s} + \sum_{n \in R_N} f(n)n^{-s},$$

for any N.

Taking the limit as $N \to \infty$ in the last inequality then gives

$$\exp(M) \geq \prod_{p} (1 + a(p)) = \prod_{p} \left(1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks} \right) = \sum_{n=1}^{\infty} f(n)n^{-s},$$

proving both of the proposition's assertions. □

We are now ready to approach the most difficult part of the proof of Dirichlet's theorem. The next Proposition will allow an obvious proof of the boundedness of $G_\chi(s)$ as $s \to 1^+$ whenever χ is nontrivial.

Proposition 4.2. Let χ be a nontrivial Dirichlet character modulo a positive integer m. Then $L_\chi(1) \neq 0$.

Proof: First, we assume that $\chi(n)$ is not real for some positive integer n. Let χ' be the character associated with χ, as in Definition 1.3. For every integer n, we have $\chi'(n + m\boldsymbol{Z})\overline{\chi'(n + m\boldsymbol{Z})} = 1$. Defining $\overline{\chi'}$ by $\overline{\chi'}(n+m\boldsymbol{Z}) = \overline{\chi'(n + m\boldsymbol{Z})}$ for every n gives $\overline{\chi'} = (\chi')^{-1}$. As noted in Definition 1.1, $\widehat{U(\boldsymbol{Z}_m)}$ is a group, so in particular, $\overline{\chi'} \in \widehat{U(\boldsymbol{Z}_m)}$. Let $\overline{\chi}$ be the Dirichlet character modulo m with associated character $\overline{\chi'}$. Then clearly, $\overline{\chi}(n) = \overline{\chi(n)}$ for every positive integer n (in particular, $\overline{\chi}$ is nontrivial). Since $\chi(n)$ is not real for some n, we have $\overline{\chi} \neq \chi$. By definition, for real $s > 1$,

$$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)n^{-s},$$

so that

$$\overline{L_\chi(s)} = \sum_{n=1}^{\infty} \overline{\chi(n)}n^{-s} = \sum_{n=1}^{\infty} \overline{\chi}(n)n^{-s} = L_{\overline{\chi}}(s).$$

Suppose $L_\chi(1) = 0$. Since both L_χ and $L_{\overline{\chi}}$ are analytic whenever $\Re(s) > 0$ (Proposition 3.2),

$$L_{\overline{\chi}}(1) = \lim_{s \to 1^+} \overline{L_\chi(s)} = 0,$$

as well. Consider the product $\prod_{\eta \in \Delta_m} L_\eta(s)$ as $s \to 1^+$. By the corollary to Proposition 3.1, L_{χ_0} has a simple pole at 1 and is analytic elsewhere in some neighborhood of 1. By

Proposition 3.2, every other factor is analytic at $s = 1$. The argument above shows that the two factors L_χ and $L_{\overline{\chi}}$ have zeros at $s = 1$. Overall, this gives exactly one factor with a simple pole at 1, and at least two factors with zeros at 1, so that

$$\lim_{s\to 1^+} \prod_{\eta\in\Delta_m} L_\eta(s) = 0.$$

However, by Lemma 4.1, $\prod_{\eta\in\Delta_m} L_\eta(s) > 1$ for all real $s > 1$. Since the product is analytic for s in a neighborhood of $s = 1$, this is a contradiction to the last equality. Thus we conclude $L_\chi(1) \neq 0$.

It remains to consider the case in which χ is real for all arguments. In this case, for every positive integer n, we have $\chi(n) = 1$ or $\chi(n) = -1$. Assume $L_\chi(1) = 0$. Whenever $\Re(s) > 1$ we have $\zeta(s) \neq 0$ (Corollary to Proposition 2.3). Then by Lemma 3.3, we also have $L_{\chi_0}(s) \neq 0$ whenever $\Re(s) > 1$. By the corollary to Proposition 3.1, $L_{\chi_0}(s)$ is analytic whenever $\Re(s) > 0$ and $s \neq 1$, and has a simple pole at $s = 1$. By Proposition 3.2, $L_\chi(s)$ is analytic whenever $\Re(s) > 0$. The assumed zero of L_χ at 1 then ensures that $L_{\chi_0}(s)L_\chi(s)$ has a removable discontinuity at $s = 1$. Thus we can define a function ψ by

$$\psi(s) = \frac{L_\chi(s)L_{\chi_0}(s)}{L_{\chi_0}(2s)},$$

analytic on $\{s \in \boldsymbol{C} \mid \Re(s) > 1/2\}$. Note that $L_\chi(s)L_{\chi_0}(s)$ is analytic whenever $s \neq 1$ and $\Re(s) > 0$. By the corollary to Proposition 3.1, $L_{\chi_0}(2s)$ has a simple pole at $s = 1/2$. Therefore, $\psi(s) \to 0$ as $s \to 1/2^+$. Furthermore, since $\psi(s)$ is analytic whenever $\Re(s) > 1/2$, we have a power series representation

$$\psi(s) = \sum_{k=0}^{\infty} b_k(s-2)^k, \tag{4}$$

valid whenever $|s-2| < \frac{3}{2}$.

We now express ψ in another form, using Proposition 4.1. For now, let $s > 1$ be real. By Theorem 3, we have the product expansions $L_\chi(s) = \prod_p (1-\chi(p)p^{-s})^{-1}$, and $L_{\chi_0}(s) = \prod_p (1-\chi_0(p)p^{-s})^{-1}$. Thus,

$$\begin{aligned}
\psi(s) &= \frac{L_\chi(s)L_{\chi_0}(s)}{L_{\chi_0}(2s)} \\
&= \frac{\left(\prod_p (1-\chi(p)p^{-s})^{-1}\right)\left(\prod_p (1-\chi_0(p)p^{-s})^{-1}\right)}{\prod_p (1-\chi_0(p)p^{-2s})^{-1}} \\
&= \prod_p \frac{(1-\chi_0(p)p^{-2s})}{(1-\chi(p)p^{-s})(1-\chi_0(p)p^{-s})}.
\end{aligned}$$

Recall that $\chi_0(p) = 1$ whenever $p \nmid m$, and $\chi_0(p) = \chi(p) = 0$ otherwise. Therefore,

$$\frac{(1-\chi_0(p)p^{-2s})}{(1-\chi(p)p^{-s})(1-\chi_0(p)p^{-s})} = \frac{(1-p^{-2s})}{(1-\chi(p)p^{-s})(1-p^{-s})}$$

if $p \nmid m$ and

$$\frac{(1-\chi_0(p)p^{-2s})}{(1-\chi(p)p^{-s})(1-\chi_0(p)p^{-s})} = 1$$

otherwise. As a result,

$$\begin{aligned}
\psi(s) &= \prod_p \frac{(1-\chi_0(p)p^{-2s})}{(1-\chi(p)p^{-s})(1-\chi_0(p)p^{-s})} \\
&= \prod_{p\nmid m} \frac{(1-p^{-2s})}{(1-\chi(p)p^{-s})(1-p^{-s})} \\
&= \prod_{p\nmid m} \frac{(1-p^{-s})(1+p^{-s})}{(1-\chi(p)p^{-s})(1-p^{-s})} \\
&= \prod_{p\nmid m} \frac{1+p^{-s}}{1-\chi(p)p^{-s}}.
\end{aligned}$$

When $\chi(p) = -1$ we have

$$\frac{1+p^{-s}}{1-\chi(p)p^{-s}} = 1.$$

On the other hand, when $\chi(p) = 1$ we have

$$\frac{1+p^{-s}}{1-\chi(p)p^{-s}} = \frac{1+p^{-s}}{1-p^{-s}}.$$

Thus,

$$\psi(s) = \prod_{p\nmid m} \frac{1+p^{-s}}{1-\chi(p)p^{-s}} = \prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}},$$

where the product is now over all primes p such that $\chi(p) = 1$.

Since $s > 1$, $p^{-s} < 1$ for every prime, and we can use the usual expression for the sum of a geometric series to get

$$\frac{1}{1-p^{-s}} = \sum_{k=0}^{\infty} p^{-ks},$$

so that

$$\begin{aligned}
\frac{1+p^{-s}}{1-p^{-s}} &= (1+p^{-s}) \sum_{k=0}^{\infty} p^{-ks} \\
&= \sum_{k=0}^{\infty} p^{-ks} + \sum_{k=0}^{\infty} p^{-(k+1)s} \\
&= 1 + 2\sum_{k=1}^{\infty} p^{-ks}.
\end{aligned}$$

Thus we find

$$\psi(s) = \prod_{\chi(p)=1} \frac{1+p^{-s}}{1-p^{-s}} = \prod_{\chi(p)=1} \left(1 + \sum_{k=1}^{\infty} 2p^{-ks}\right).$$

We are now ready to apply Proposition 4.1. Define a function f on the positive integers by the following:

- $f(p^k) = 2$ whenever p is a prime with $\chi(p) = 1$ and k is any positive integer
- $f(p^k) = 0$ whenever p is a prime with $\chi(p) \neq 1$ and k is any positive integer
- $f(1) = 1$
- $f(m)f(n) = f(mn)$ whenever $(m, n) = 1$.

Note that since every positive integer n has a unique factorization into a product of prime powers, these conditions uniquely define $f(n)$. Clearly, $0 \leq f(p^k) \leq 2$ for every prime p and positive integer k. Furthermore, if p is a prime then

$$1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks} = 1 + \sum_{k=1}^{\infty} 2p^{-ks}$$

whenever $\chi(p) = 1$ and

$$1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks} = 1$$

whenever $\chi(p) \neq 1$. Rewriting ψ in terms of f and using Proposition 4.1, we find

$$\begin{aligned}
\psi(s) &= \prod_{\chi(p)=1} \left(1 + \sum_{k=1}^{\infty} 2p^{-ks}\right) \\
&= \prod_{p} \left(1 + \sum_{k=1}^{\infty} f(p^k)p^{-ks}\right) \\
&= \sum_{n=1}^{\infty} f(n)n^{-s}.
\end{aligned}$$

Returning to (4), for real $s > 1$ with $|s - 2| < \frac{3}{2}$, we have

$$\psi(s) = \sum_{n=1}^{\infty} f(n)n^{-s} = \sum_{k=0}^{\infty} b_k(s-2)^k.$$

By Taylor's theorem (see [3, Ch.21], [4, Ch.2]), for each $k \geq 0$, $b_k = \frac{\psi^{(k)}(2)}{k!}$. To find the kth derivative of ψ, we differentiate the series $\sum_{n=1}^{\infty} f(n)n^{-s}$. Arguing inductively, for each pair of nonnegative integers k and n, we obtain

$$\frac{d^k}{ds^k} f(n)n^{-s} = f(n)(-\log n)^k n^{-s}.$$

Let $\delta > 0$, and assume $s > 1 + 2\delta$. By Lemma 4.2, there is some integer M such that for any $n \geq M$,

$$|f(n)(-\log n)^k n^{-s}| \leq f(n)n^{-s+\delta} \leq f(n)n^{-(1+2\delta)+\delta} = f(n)n^{-1-\delta}.$$

By Proposition 4.1, $\sum_{n=M}^{\infty} f(n)n^{-1-\delta}$ converges, so by the Weierstrass M-test,

$$\sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s}$$

converges absolutely for $s > 1$ and uniformly on $s \in \{x \in \boldsymbol{R} \mid x > 1 + 2\delta\}$. By a result in real analysis (see ∮20.5 of [3]), we then have

$$\sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-s} = \sum_{n=1}^{\infty} \frac{d^k}{ds^k} f(n)n^{-s} = \psi^{(k)}(s)$$

for all $s > 1$ and nonnegative integers k. This implies that

$$b_k = \frac{\psi^{(k)}(2)}{k!} = \frac{1}{k!} \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-2}$$

for each $k \geq 0$. We can let

$$c_k = \frac{1}{k!} \sum_{n=1}^{\infty} f(n)(\log n)^k n^{-2} \geq 0,$$

so that

$$b_k = (-1)^k c_k.$$

In this way,

$$\psi(s) = \sum_{k=0}^{\infty} b_k(s-2)^k = \sum_{k=0}^{\infty} c_k(2-s)^k$$

whenever $|s - 2| < 3/2$. For $1/2 \leq s \leq 2$, all of the terms in the series on the right are nonnegative. Furthermore, we have

$$c_0 = \frac{1}{0!} \sum_{n=1}^{\infty} f(n)(\log n)^0 n^{-2} \geq f(1) = 1.$$

Thus, for $1/2 \leq s \leq 2$,

$$\psi(s) = \sum_{k=0}^{\infty} c_k(2-s)^k \geq c_0(2-s)^0 = c_0 \geq 1.$$

We have found that $\psi(s)$ is differentiable for $s > 1/2$, that $\lim_{s\to 1/2^+} \psi(s) = 0$, and that $\psi(s) \geq 1$ for $1/2 \leq s \leq 2$. This is a contradiction, and so we conclude that $L_\chi(1) \neq 0$, as we wanted. □

We are now prepared to examine the behaviour of $G_\chi(s)$ as $s \to 1^+$. We have already determined the behaviour of L_χ at 1. As G_χ is essentially the logarithm of L_χ, this will be sufficient.

Proposition 4.3.

$$\lim_{s\to 1^+} \frac{G_{\chi_0}(s)}{\log\left[(s-1)^{-1}\right]} = 1.$$

Furthermore, if $\chi \in \Delta_m$ is nontrivial then $G_\chi(s)$ is bounded as $s \to 1^+$.

Proof: Assume $s > 1$. By definition,

$$G_{\chi_0}(s) = \sum_p \sum_{k=1}^{\infty} (1/k)\chi_0(p^k)p^{-ks} = \sum_{p\nmid m} \sum_{k=1}^{\infty} (1/k)p^{-ks},$$

so $G_{\chi_0}(s)$ is clearly real. By Proposition 2.3, $\exp(G_{\chi_0}(s)) = L_{\chi_0}(s)$. Thus we must have $G_{\chi_0}(s) = \log(L_{\chi_0}(s))$ (recall that log denotes the real natural logarithm). By Lemma 3.3,

$$L_{\chi_0}(s) = \prod_{p|m} (1-p^{-s})\zeta(s),$$

so that

$$G_{\chi_0}(s) = \log(L_{\chi_0}(s)) = \sum_{p|m} \log(1-p^{-s}) + \log(\zeta(s)),$$

and

$$\frac{G_{\chi_0}(s)}{\log\left[(s-1)^{-1}\right]} = \frac{\sum_{p|m} \log(1-p^{-s}) + \log(\zeta(s))}{\log\left[(s-1)^{-1}\right]}.$$

Clearly $\sum_{p|m} \log(1-p^{-s})$ is bounded as $s \to 1^+$. Observe that $\lim_{s\to 1^+} \log\left[(s-1)^{-1}\right] = \infty$. Thus we find that

$$\begin{aligned} \lim_{s\to 1^+} \frac{G_{\chi_0}(s)}{\log\left[(s-1)^{-1}\right]} &= \lim_{s\to 1^+} \frac{\sum_{p|m} \log(1-p^{-s}) + \log(\zeta(s))}{\log\left[(s-1)^{-1}\right]} \\ &= \lim_{s\to 1^+} \frac{\log(\zeta(s))}{\log\left[(s-1)^{-1}\right]}. \end{aligned}$$

By the corollary to Proposition 2.2, the last limit is equal to 1, and we find

$$\lim_{s\to 1^+} \frac{G_{\chi_0}(s)}{\log\left[(s-1)^{-1}\right]} = 1,$$

as we wanted.

Now let $\chi \in \Delta_m$ be nontrivial. By Proposition 2.3,

$$\exp(G_\chi(s)) = L_\chi(s).$$

By Proposition 2.4, G_χ is continuous on $(1,\infty)$, and by Proposition 3.2, L_χ is analytic on $\{z \in \boldsymbol{C} \mid \Re z > 0\}$. By Proposition 4.2, $L_\chi(1) \neq 0$, so that $\operatorname{Log}(L_\chi(1))$ is defined. The exponential map is continuous, so we find that

$$\exp(\lim_{s\to 1^+} G_\chi(s)) = \lim_{s\to 1^+} \exp(G_\chi(s)) = \lim_{s\to 1^+} L_\chi(s) = L_\chi(1) \neq 0.$$

As a result,

$$\lim_{s\to 1^+} G_\chi(s) = \operatorname{Log}(L_\chi(1)) + 2\pi i k$$

for some integer k. In particular, $G_\chi(s)$ is bounded as $s \to 1^+$, as we wanted. □

VI. CONCLUSION TO THE PROOF OF DIRICHLET'S THEOREM

Let m be a positive integer, and let $s > 1$ be real. By Proposition 2.5, for $\chi \in \Delta_m$, we have

$$G_\chi(s) = \sum_{p\nmid m} \chi(p)p^{-s} + R_\chi(s),$$

where $R_\chi(s)$ is bounded as $s \to 1^+$. Let a be an integer with $(a,m) = 1$. Then for each Dirichlet character χ,

$$\overline{\chi(a)}G_\chi(s) = \sum_{p\nmid m} \overline{\chi(a)}\chi(p)p^{-s} + \overline{\chi(a)}R_\chi(s).$$

Taking the sum over all Dirichlet characters, we find

$$\sum_{\chi\in\Delta_m} \overline{\chi(a)}G_\chi(s) = \sum_{\chi\in\Delta_m} \left(\sum_{p\nmid m} \overline{\chi(a)}\chi(p)p^{-s}\right) + \sum_{\chi\in\Delta_m} \overline{\chi(a)}R_\chi(s).$$

Here, $\sum_{p\nmid m} |\chi(p)p^{-s}|$ is bounded by $\sum_n n^{-s} = \zeta(s)$ and thus $\sum_{p\nmid m} \chi(p)p^{-s}$ is absolutely convergent. As a result, we can rearrange terms to find

$$\sum_{\chi\in\Delta_m} \overline{\chi(a)}G_\chi(s) = \sum_{p\nmid m}\left(p^{-s}\sum_{\chi\in\Delta_m}\chi(p)\overline{\chi(a)}\right) + \sum_{\chi\in\Delta_m}\overline{\chi(a)}R_\chi(s).$$

Here $(a, m) = 1$ by assumption. Thus by Corollary 1.2.1, if $p \nmid m$ (so $(p, m) = 1$), then $\sum_{\chi\in\Delta_m} \chi(p)\overline{\chi(a)}$ is equal to $\phi(m)$ when $p \equiv a \pmod{m}$ and is equal to 0 otherwise. We use this to simplify our last equality to

$$\sum_{\chi\in\Delta_m} \overline{\chi(a)}G_\chi(s) = \phi(m)\sum_{p\in\mathcal{P}(a;m)} p^{-s} + \sum_{\chi\in\Delta_m}\overline{\chi(a)}R_\chi(s),$$

where we have used the definition of $\mathcal{P}(a; m)$ from the statement of Theorem 4 ($\mathcal{P}(a; m)$ is the set of all primes congruent to a modulo m). Let $\mathcal{G}_a(s) = \sum_\chi \overline{\chi(a)}G_\chi(s)$, where the sum is over all nontrivial Dirichlet characters, and let $\mathcal{R}_a(s) = \sum_{\chi\in\Delta_m}\overline{\chi(a)}R_\chi(s)$. We rearrange our previous equality to find

$$\overline{\chi_0(a)}G_{\chi_0}(s) + \mathcal{G}_a(s) = \phi(m)\sum_{p\in\mathcal{P}(a;m)} p^{-s} + \mathcal{R}_a(s).$$

Dividing by $\log\left[(s-1)^{-1}\right]$ and taking the limit as $s \to 1^+$, we find

$$\lim_{s\to1^+}\frac{\overline{\chi_0(a)}G_{\chi_0}(s) + \mathcal{G}_a(s)}{\log\left[(s-1)^{-1}\right]} = \lim_{s\to1^+}\frac{\phi(m)\sum_{p\in\mathcal{P}(a;m)} p^{-s} + \mathcal{R}_a(s)}{\log\left[(s-1)^{-1}\right]}, \tag{5}$$

if the limits exist. By Proposition 4.3,

$$\lim_{s\to1^+}\frac{\overline{\chi_0(a)}G_{\chi_0}(s)}{\log\left[(s-1)^{-1}\right]} = \overline{\chi_0(a)}.$$

Again by Proposition 4.3, $G_\chi(s)$ is bounded as $s \to 1^+$ whenever χ is nontrivial, so that $\mathcal{G}_a(s)$ is bounded as $s \to 1^+$. Recall that $R_\chi(s)$ is bounded as $s \to 1^+$ for each character χ, so that $\mathcal{R}_a(s)$ is as well. Finally, observe that $\lim_{s\to1^+}\log\left[(s-1)^{-1}\right] = \infty$. Thus

$$\lim_{s\to1^+}\frac{\mathcal{G}_a(s)}{\log\left[(s-1)^{-1}\right]} = \lim_{s\to1^+}\frac{\mathcal{R}_a(s)}{\log\left[(s-1)^{-1}\right]} = 0.$$

Using these to simplify (5), we find

$$\begin{aligned}
\overline{\chi_0(a)} &= \lim_{s\to1^+}\frac{\overline{\chi_0(a)}G_{\chi_0}(s) + \mathcal{G}_a(s)}{\log\left[(s-1)^{-1}\right]} \\
&= \lim_{s\to1^+}\frac{\phi(m)\sum_{p\in\mathcal{P}(a;m)} p^{-s} + \mathcal{R}_a(s)}{\log\left[(s-1)^{-1}\right]} \\
&= \phi(m)\lim_{s\to1^+}\frac{\sum_{p\in\mathcal{P}(a;m)} p^{-s}}{\log\left[(s-1)^{-1}\right]},
\end{aligned}$$

or

$$\frac{\overline{\chi_0(a)}}{\phi(m)} = \lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}(a;m)} p^{-s}}{\log\left[(s-1)^{-1}\right]}.$$

Here we have $(a,m) = 1$ by assumption, so by definition, $\chi_0(a) = 1$. Furthermore, from Definition 3.1,

$$\lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}(a;m)} p^{-s}}{\log\left[(s-1)^{-1}\right]} = d(\mathcal{P}(a;m)),$$

if the limit exists. Thus

$$\frac{1}{\phi(m)} = \frac{\overline{\chi(a)}}{\phi(m)} = \lim_{s\to 1^+} \frac{\sum_{p\in\mathcal{P}(a;m)} p^{-s}}{\log\left[(s-1)^{-1}\right]} = d(\mathcal{P}(a;m)),$$

as asserted in the statement of Theorem 4 (Dirichlet's theorem). In particular, since $\mathcal{P}(a;m)$ has nonzero Dirichlet density, by Lemma 3.1, it is infinite. □

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