Properties of Ramanujan Graphs

Andrew Droll[1,*]

[1]*Department of Mathematics and Statistics, Jeffery Hall, Queen's University*

Kingston, Ontario, Canada

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[*]Electronic address: adroll@mast.queensu.ca

I. INTRODUCTION

Graph theory has wide-ranging real-world applications while at the same time maintains its status as one of mathematics' most prominently studied fields. One of the more recently considered topics in graph theory is that of Ramanujan Graphs; in short, this amounts to the study of infinite families of k-regular graphs with minimal upper bounds on their non-trivial eigenvalues. It turns out that such graphs have small diameter and also make good expanders. Thus this topic is of some interest to the theory of communication networks. Another, more pedagogical source of interest from this topic is the fact that its study intertwines numerous branches of mathematics, from number theory to group theory, algebraic geometry, and representation theory.

In this paper we examine some ways to construct Ramanujan graphs based on recent additions to the literature, as well as examining some of their properties. More precisely, we will first motivate the definition of a Ramanujan graph with some examples and simple proofs. We will furthermore examine the construction of k-regular graphs as Cayley graphs of Abelian groups and give some examples that result in Ramanujan graphs. Despite the fact that we can construct examples of Ramanujan graphs in this way, we will show that it is impossible to construct infinite families of Ramanujan graphs using Cayley graphs in nontrivial cases. Finally, we will prove some deeper results on the motivation for studying Ramanujan graphs and also present the notion of the zeta function of a graph. The paper will roughly follow the paper *Ramanujan Graphs* by Ram Murty [11], with more detail on abelian Cayley graphs derived in large part from [12].

II. SOME DEFINITIONS

Here we will define some basic notions concerning graphs that will be relevant throughout the paper. In the next section we will prove some simple properties following from these definitions.

Definition 1.1. A *graph* consists of a nonempty set V of vertices and a set E of edges, along with a function $f : E \to V \times V$ associating each edge to a pair of vertices (where the order of this pair does not matter). If $f(e) = (v_1, v_2)$ then v_1 and v_2 are the endpoints of the edge e. A *simple graph* is a graph in which each edge is associated to a pair of distinct vertices, and no two edges are associated with the same pair of vertices.

If X is a graph, we denote the vertex set of X by $V(X)$ and the edge set of X by $E(X)$. A *bipartite* graph is one whose vertex set can be partitioned into two disjoint sets A and B, such that the pair associated to any edge contains one vertex of A and one vertex of B (i.e. no two vertices of A or of B are joined by an edge).

Definition 1.2. Let X be a graph, and e be an edge of X. The *endpoints* of e are the two vertices in the pair associated to e. The edge e is called a *loop* if both endpoints are the same vertex. Two vertices are called *adjacent* if they are both endpoints of the same edge. If v is an endpoint of the edge e, then e is said to be *incident* with v.

Definition 1.3. Let X be a simple graph. An *adjacency matrix* A_X of X is a matrix whose rows and columns are indexed by the vertices of X, and where the (i, j)th entry of A_X is the number of edges in $E(X)$ associated to the vertex pair (i, j), i.e. the number of edges joining vertex i to vertex j. The *degree* $\deg(i)$ of the vertex i is the sum over all entries in the ith row of A_X, which is also equal to the number of edges incident with vertex i.

If X is not simple, we define the adjacency matrix in the same way. However, the degree of a vertex is instead defined to be the number of edges incident with the vertex, where loops are counted twice. It is also important to note that if A_X is any adjacency matrix of X, then it is similar to all other adjacency matrices of X by some permutation matrix. Furthermore, an adjacency matrix A_X of a graph X is necessarily symmetric (this is no longer true if we consider directed graphs).

We refer to the set of eigenvalues of A_X along with their multiplicities as the spectrum of the graph X. Note that since all possibilities for A_X are similar to one another (depending on the indexing of the vertices of X), it is clear from linear algebra that the eigenvalues are the same for any choice of A_X and so the spectrum of X is well-defined.

Definition 1.4. A *walk* from v_0 to v_n in a graph X is a sequence $v_0, ..., v_n$ where $v_i \in V(X)$ for all i, and v_i is adjacent to v_{i+1} for all i. A *path* is a walk in which no vertex is repeated. The *length* of an edge is defined to be 1. The length of a walk is the sum of the lengths of edges joining its vertices.

Definition 1.5. The *distance* between two vertices x and y in a graph X, $d(x, y)$, is the minimal length over all paths from x to y. If no such path exists then the distance is defined to be infinite. The graph X is called *connected* if $d(x, y)$ is finite for every pair of vertices x, y.

Definition 1.6. A graph X is called k-regular if every vertex of X has degree k.

Ramanujan graphs (defined below) are k-regular for some k.

Definition 1.7. Let X be a graph. A *subgraph* of X is a graph Y whose vertex set $V(Y)$ is a subset of $V(X)$ and whose edge set $E(Y)$ is a subset of $E(X)$, such that we necessarily have all endpoints of elements of $E(Y)$ in $V(Y)$ (this is part of the definition of a graph, after all). A *proper subgraph* of X is a subgraph which is not equal to X.

Definition 1.8. Let X be a graph. A *connected component* of X is a subgraph Y of X such that Y is connected, every edge of $E(X)$ whose endpoints are both in $V(Y)$ is in $E(Y)$, and if Z is a subgraph of X such that Y is a proper subgraph of Z, then Z is not connected.

Every graph can be partitioned uniquely into some number of connected components.

III. BASIC PROPERTIES OF k-REGULAR GRAPHS

Here we examine some elementary properties of k-regular graphs, which will help motivate the study of Ramanujan graphs to follow. Throughout this section, let X be a graph.

Our first lemma in this section is a very basic result in graph theory, which we restate only for convenience.

Lemma 2.1. We have

$$\sum_{v \in V(X)} \deg(v) = 2|E(X)|. \tag{1}$$

Proof: Each edge which is not a loop is counted in the degrees of exactly two vertices (its endpoint vertices). Each loop is counted twice in the degree of its endpoint vertex. □

The next result is again very basic, but quite important in several arguments to follow.

Lemma 2.2. Let A_X be an adjacency matrix for X, with columns and rows indexed by the vertices of X. Then the (i, j)th entry of A_X^r is the number of walks of length r from vertex i to vertex j.

Proof: The (i, j)th entry of A_X is the number of walks of length 1 from i to j by definition. Proceeding by induction, we suppose A_X^r counts the numer of walks of length r from x to y in its (x, y)th entry. The definition of matrix multiplication makes the result clear (it is precisely the combinatorial computation of counting how many ways there are of adding a walk of length 1 to a walk of length r to get from x to y).□

Lemma 2.2 will be particularly important when we motivate the study of Ramanujan graphs, and in the last section when we give a brief presentation of the zeta function of a graph.

Theorem 2.1. Let A_X be an adjacency matrix for X and let $\Delta(X)$ denote the largest degree of any vertex of X. Then, if λ is an eigenvalue of A_X, we have $|\lambda| \leq \Delta(X)$.

Proof: Suppose $|V(X)| = n$ and $v = (x_1, ...x_n)^t$ is an eigenvector of A_X corresponding to the eigenvalue λ. Suppose without loss of generality that $|x_1| \geq |x_i|$ for all i, $1 \leq i \leq n$ (otherwise, re-index the vertices so that this is true). The first entry of $A_X v$ is λx_1 by definition of an eigenvector. However, if $a_{i,j}$ denotes the (i, j)th entry of A_X, then by definition of matrix multiplication, it is also equal to

$$\sum_{i=1}^{n} a_{1,j} x_j.$$

It follows that

$$|\lambda x_1| = \Big|\sum_{i=1}^{n} a_{1,i} x_i\Big| \leq \sum_{i=1}^{n} |a_{1,i}||x_i| \leq |x_1| \sum_{i=1}^{n} |a_{1,i}|.$$

Here, $\sum_{i=1}^{n} |a_{1,i}|$ is precisely the degree of vertex 1, by definition of A_X. Thus

$$|\lambda||x_1| \leq |x_1| \deg(1) \leq |x_1| \Delta(X),$$

by definition of $\Delta(X)$, and the result follows.□

The next two results are basic properties of eigenvalues of k-regular graphs, which will be quite useful in the rest of the paper.

Proposition 2.1. If X is k-regular, then k is an eigenvalue of A_X and all eigenvalues have absolute value of at most k.

Proof: (1,...,1) is an eigenvector with eigenvalue k, so k is an eigenvalue. Since every vertex has degree k by definition, Theorem 2.1 gives the second part of the statement.□

Recall that the algebraic multiplicity of an eigenvalue λ of a matrix is the number of times that $(x - \lambda)$ appears in the factorization of the characteristic determinant. On the other hand, the geometric multiplicity of λ is the dimension of the space of vectors for which λ is an eigenvalue.

Proposition 2.2. If X is k-regular, then the algebraic and geometric multiplicities of the eigenvalue k of A_X are equal to the number of connected components of X.

Proof: By Proposition 2.1, k is an eigenvalue. Let the connected components of X be $X_1, ..., X_p$. Suppose v is any eigenvector with eigenvalue k, and write $v = (x_1, ..., x_n)^t$. As in the proof of Proposition 2.1, suppose without loss of generality that $|x_1| \geq |x_i|$ for all i,

$1 \leq i \leq n$ (otherwise, re-index the vertices), and $x_1 > 0$. As in the proof of Proposition 2.1, we have

$$kx_1 = \sum_{i=1}^{n} a_{1,i}x_i \leq x_1 \sum_{i=1}^{n} a_{1,i} = x_1 \deg(1).$$

Here, $\deg(1)$ is just k, since X is k-regular by assumption. Thus we see

$$\sum_{i=1}^{n} a_{1,i}x_i = x_1 \sum_{i=1}^{n} a_{1,i}.$$

As $|x_i| \leq x_1$ for all i by assumption, this means $x_i = x_1$ for all i with $a_{1,i} \neq 0$, i.e. for all i such that vertex i is adjacent to vertex 1. We can then iterate this argument for all of the vertices adjacent to vertex 1, and continue in this manner to show that $x_j = x_1$ whenever vertex i is connected to vertex 1 by a path (i.e. when $d(i, 1)$ is finite). Furthermore, we can apply this argument to every connected component of X.

We have thus shown that any eigenvector $(y_1, ..., y_n)$ has $y_i = y_j$ whenever vertices i and j are connected by a path. If we define v_j to be the vector having 1's in all positions corresponding to vertices in X_j and 0's elsewhere, we can easily see that v_j is an eigenvector with eigenvalue k using matrix multiplication. Furthermore by our argument above, every eigenvector with eigenvalue k must be a linear combination of $v_1, ..., v_p$, and clearly $v_1, ..., v_p$ are linearly independent. Thus the (geometric) multiplicity of the eigenvalue k is precisely p, the number of connected components of X. Since A_X is symmetric, it is diagonalizable by results from linear algebra, and the geometric multiplicity is equal to the algebraic multiplicity. $\square$

We also note a useful result on bipartite graphs without proof (See [16]):

Proposition 2.3. If X is bipartite and λ is an eigenvalue of A_X multiplicity of m, then $-\lambda$ is as well. Furthermore, if $\lambda_0(X)$ is the largest eigenvalue of X, and X is connected, then $-\lambda_0(X)$ is an eigenvalue only if X is bipartite.

We can now proceed to motivate the definition of a Ramanujan graph, and give some examples.

IV. RAMANUJAN GRAPHS

We will begin by providing some preliminary motivation for the definition of a Ramanujan graph, and then state the definition itself. We will reserve examples for the next section, in which we will introduce some of the theory of Cayley graphs of abelian groups and use it to prove that certain particular graphs are Ramanujan.

The following is a result by Chung [1] which will serve as a starting point for the motivation for our definition of a Ramanujan graph. Throughout this section, let X be a k-regular graph. We will also assume that X is simple unless otherwise specified.

Definition 3.1. The *diameter* of a connected graph X is defined to be $\operatorname{diam}(X) = \max_{v,w\in V(X)}(d(v,w))$.

In addition, we make the following definition for convenience.

Definition 3.2. If X is any k-regular graph (not necessarily simple) and $|V(X)| \geq 3$ we define $\lambda(X)$ to be the maximum absolute value of all eigenvalues of A_X between $-k$ and k (i.e. $\lambda(X) < k$ by definition).

We are now prepared to state and prove Chung's result.

Theorem 3.1. Let $n = |V(X)|$ (the number of vertices of X). If X is not bipartite, then

$$\operatorname{diam}(X) \leq \frac{\log(n-1)}{\log(k/\lambda(X))} + 1.$$

If X is bipartite,

$$\operatorname{diam}(X) \leq \frac{\log(n-2)}{\log(k/\lambda(X))} + 2.$$

Proof: As A_X is symmetric, it is orthogonally diagonalizable by results in linear algebra. Let $u_0, ..., u_{n-1}$ be an orthonormal eigenbasis. Assume without loss of generality that $u_0 = \sqrt{\frac{1}{k}}(1, 1, .., 1)^t$ with eigenvalue k (as we saw in the proof of Proposition 2.1). Furthermore let $k = \lambda_0, \lambda_1, ..., \lambda_{n-1}$ be the eigenvalues corresponding to $u_0, ..., u_{n-1}$.

Writing $U = [u_0, ..., u_{n-1}]$, we have $U^t = U^{-1}$ and

$$UDU^t = A_X$$

where $D = \mathrm{diag}(\lambda_0, \lambda_1, ..., \lambda_{n-1})$. Furthermore, this shows

$$UD^rU = A_X^r$$

for $r \in \mathbb{N}$, where $D^r = \mathrm{diag}(\lambda_0^r, \lambda_1^r, ..., \lambda_{n-1}^r)$. Using the definition of matrix multiplication, we can re-write this as

$$A_X^r = \sum_{i=0}^{n-1} \lambda_i^r u_i u_i^t,$$

or

$$(A_X^r)_{x,y} = \sum_{i=0}^{n-1} \lambda_i^r (u_i u_i^t)_{x,y}.$$

Using our construction of u_0 and λ_0, we see that $(u_0 u_0^t)_{x,y} = \frac{1}{n}$, and noting that $(u_i u_i^t)_{x,y} = (u_i)_x (u_i)_y$, we see that

$$(A_X^r)_{x,y} = \frac{k^r}{n} + \sum_{i=1}^{n-1} \lambda_i^r (u_i)_x (u_i)_y \geq \frac{k^r}{n} - \Big| \sum_{i=1}^{n-1} \lambda_i^r (u_i)_x (u_i)_y \Big|. \tag{2}$$

If X is not bipartite, then $|\lambda_i| \leq \lambda(X)$ for all $i \geq 1$ and using the well-known Cauchy-Schwarz inequality, we get

$$\Big| \sum_{i=1}^{n-1} \lambda_i^r (u_i)_x (u_i)_y \Big| \leq \lambda(X)^r \sum_{i=1}^{n-1} |(u_i)_x (u_i)_y| \leq \lambda(X)^r \left(\sum_{i=1}^{n-1} (u_i)_x^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{n-1} (u_i)_y^2 \right)^{\frac{1}{2}}. \tag{3}$$

Now, as the columns of U are orthonormal vectors, it is an orthogonal matrix and its rows are necessarily orthonormal as well. In particular, we must have

$$\left(\sum_{i=0}^{n-1} (u_i)_x^2 \right)^{\frac{1}{2}} = 1,$$

or

$$\left(\sum_{i=1}^{n-1} (u_i)_x^2 \right) = 1 - (u_0)_x^2 = 1 - \frac{1}{n}.$$

Thus (3) says

$$\Big|\sum_{i=1}^{n-1} \lambda_i (u_i)_x (u_i)_y\Big| \leq \lambda(X)^r \left(1 - \frac{1}{n}\right)$$

and (2) gives that

$$(A_X^r)_{x,y} \geq \frac{k^r}{n} - \lambda(X)^r \left(1 - \frac{1}{n}\right).$$

Thus, provided that

$$\frac{k^r}{n} > \lambda(X)^r \left(1 - \frac{1}{n}\right),$$

which is equivalent to

$$\frac{k^r}{\lambda(X)^r} > n - 1,$$

every entry of A_X^r is positive. If every entry of A_X^r is positive then there is a path of length r between each two vertices of X, and so $\mathrm{diam}(x) \leq r$. Here,

$$\frac{k^r}{\lambda(X)^r} > n - 1$$

means

$$r(\log(k/\lambda(X))) > \log(n-1),$$

or

$$r > \frac{\log(n-1)}{\log(k/\lambda(X))}.$$

As the diameter is less than or equal to any r satisfying this condition, it follows that

$$\mathrm{diam}(X) \leq \frac{\log(n-1)}{\log(k/\lambda(X))} + 1,$$

as we wanted.

The bipartite case is somewhat different, because no power of A_X has all positive entries. This difficulty may be resolved by noting that if $r = \mathrm{diam}(X)$, then r is the smallest nonnegative integer such that $I + ... + A_X^r$ has all positive entries, and proceeding with a similar argument.□

Now, if $\lambda(X)$ is small, then $\log(k/\lambda(X))$ is large and the upper bound obtained on our diameter is small. Thus, if we wish to obtain graphs with small diameters, it is desirable to make $\lambda(X)$ as small as possible (recall $\lambda(X)$ is the maximal absolute value of eigenvalues of A_X other than $-k$ and k).

Next, we look at how small $\lambda(X)$ can realistically be taken to be for large graphs.

Proposition 3.1. Let $\{X_n\}_{n=1}^{\infty}$ be an infinite family of k-regular graphs with $|X_n| = n$. Then

$$\text{liminf}_{n\to\infty}\lambda(X_n) \geq \sqrt{k}.$$

Proof: As A_X is symmetric for any graph X, the eigenvalues of $A(X)A(X)^t = A(X)^2$ are just the squares of the eigenvalues of $A(X)$, and thus the trace is the sum of the squares of the eigenvalues of $A(X)$. Suppose $|V(X)| = n$. For a k-regular graph, we also have that the trace of $A(X)^2$ is just equal to kn (since there are k walks of length 2 between each vertex and itself: following one of the k edges out and the same edge back is the only way to accomplish it). As all the eigenvalues other than k are smaller than $\lambda(X)$, this means if X is not bipartite then

$$k^2 + \lambda(X)^2(n-1) \geq kn,$$

or

$$\lambda(X) \geq \left(\frac{n-k}{n-1}\right)^{\frac{1}{2}}\sqrt{k}.$$

If X is bipartite, then instead there are two eigenvalues of magnitude k and we instead have

$$2k^2 + (n-2)\lambda(X)^2 \geq nk,$$

and get the similar result that

$$\lambda(X) \geq \left(\frac{n-2k}{n-2}\right)^{\frac{1}{2}}\sqrt{k}.$$

In both cases, taking the limit as $|V(X)| = n \to \infty$, the proposition clearly follows. $\square$

The result of Proposition 3.1 is not quite as strong as we would like. In fact, there have been stronger results proved by Alon and Boppana, (the second result is likely one of Alon under the alias of Nilli) which we state below and will investigate more later ([8],[14]).

Theorem 3.2.

$$\mathrm{liminf}_{\mathrm{n}\to\infty}\lambda(X_{n,k}) \geq 2\sqrt{k-1},$$

where the lim inf is taken over k-regular graphs with $|V(X_{n,k})| = n \to \infty$.

Theorem 3.3. If $\mathrm{diam}(X) \geq 2b+2 \geq 4$ for some natural b, then

$$\lambda(X) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

In fact, we can see that Theorem 3.2 is actually a corollary of Theorem 3.3 by an easy argument.

Proposition 3.2. If $\{X_n\}_{n=0}^{\infty}$ is a family of simple k-regular graphs with $|X_n| = n$, then $\mathrm{diam}(X_n) \to \infty$ as $n \to \infty$.

Proof: Given any vertex of a simple k-regular graph X, the number of distinct walks of length r starting at the vertex is certainly at most k^r, since each vertex has degree k. If the diameter of X is m, then every vertex of X can be reached by a path of length at most m from any given vertex. Since every walk of length less than m is a part of a walk of length m, we conclude that every vertex is in some walk of length m departing from any given fixed vertex. A walk of length m has at most $m+1$ distinct vertices, and as mentioned above, there are at most k^m such walks. Thus the number of vertices of X must necessarily satisfy

$$|X| \leq (m+1)k^m.$$

It follows that if $|X| \to \infty$ then $\mathrm{diam}(X) = m \to \infty$. Rephrasing this in the language of the proposition gives the result. □

This result shows that as the number of vertices in our graphs gets larger, we can take arbitrarily large b in the statement of Theorem 3.3, from which Theorem 3.2 follows as a corollary, as we claimed.

We see from Theorem 3.2 that for an infinite family of k-regular graphs we will necessarily have $\lambda(X)$ at least approaching $2\sqrt{k-1}$ (though we should note that the limit need not exist for an arbitrary family, of course) as we take graphs with more and more vertices. In fact, it is possible to construct families of graphs with exactly this limit for $\lambda(X)$ as the number of vertices get large in the case $k = p^\alpha + 1$ for some prime p and natural α (see [8], [9], [10]). The construction of infinite families of such graphs in other cases is an open problem (for example, for $k = 7$). Minimizing $\lambda(X)$ in such families gives small maximal bounds on the diameters as we saw in Theorem 3.1, which is of importance to communication networks for example. These are our initial motivations for the definition of a Ramanujan graph.

Definition 3.2. A k-regular graph X is called *Ramanujan* if $\lambda(X) \leq 2\sqrt{k-1}$.

Our interest is largely in finding infinite families of k-regular Ramanujan graphs for fixed k. We will see particular examples of Ramanujan graphs in the next section via Cayley graphs of abelian groups. Many of the simplest graphs are Ramanujan (cyclic graphs and the complete graph, for example). However, we will also prove that it is impossible to construct infinite families of k-regular Ramanujan graphs via these Cayley graphs (for fixed k).

V. CAYLEY GRAPHS

We certainly wish to find some examples of the Ramanujan graphs defined in the previous section. As we discussed before, the definition's upper bound on the graph's nontrivial eigenvalues is minimal if we consider looking at infinite families of graphs. This motivates the definition, but we have not yet seen any examples of such graphs. In this section, we define Cayley graphs of a group and find a simple expression for their eigenvalues. This will allow us to give some simple examples of Ramanujan graphs, and easily prove that they are Ramanujan.

On the other hand, we will also show in this paper that it is impossible to construct infinite families of k-regular Ramanujan graphs for fixed k via Cayley graphs of abelian groups. Thus, while we can see some particular examples of interest, more work is required in order to find the infinite families in which we are interested.

In this section, let G be a finite abelian group.

Definition 4.1. Let S be a symmetric multiset of G (that is, a subset in which elements may be repeated more than once, and which has the property that whenever $x \in S$, we also have $x^{-1} \in S$ where x^{-1} is the inverse of x in G, with the same multiplicity as x). We define the *Cayley graph* $X(G,S)$ to be a graph with vertex set equal to the set of elements of G and where the number of edges joining vertex a to vertex b in $X(G,S)$ is equal to the multiplicity of ab^{-1} in S (here, if ab^{-1} is not in S then there are no edges joining a to b).

Note that the Cayley graph $X(G,S)$ is clearly $|S|$-regular. We now state and prove a simple formula for the spectrum of $X(G,S)$.

Theorem 4.1. Let S be a symmetric multiset of G, and let $|S| = n$. Then the multiset of eigenvalues of $X(G,S)$ is

$$\{\lambda_\chi = \sum_{s \in S} \chi(s) \mid \chi \text{ is an irreducible character of } X(G,S)\}.$$

Proof: For each $g \in G$ denote the multiplicity of g in S by $n_S(g)$ (where $n_S(g) = 0$ if g is not in S). As G is abelian, every irreducible representation is a character and there are exactly $|G|$ of them (since every element of G is the only element of its conjugacy class in G. See [3]).

Now, if the elements of G are given some ordering $g_1, ..., g_n$, then for each irreducible character χ of G, define a vector v_χ in $\mathbb{R}^{|G|}$ by $v_\chi = (\chi(g_1), ..., \chi(g_n))^t$. Let $A = A_{X(G,S)}$ where the vertices of $X(G,S)$ are indexed as per the ordering on the elements of G specified above. Here, the (i,j)th entry of A is, by definition of the adjacency matrix and the general construction of a Cayley graph specified in Definition 4.1, $n_S(g_i g_j^{-1})$, i.e. the number of edges connecting g_i to g_j in $X(G,S)$.

By definition of matrix multiplication and construction of v_χ, we then have

$$(Av_\chi)_i = \sum_{g \in G} n_S(g_i^{-1}g)\chi(g).$$

We can re-index this summation by taking $g_i^{-1}g = x$ (since it will still go over all elements of G, just in a different order) to get

$$(Av_\chi)_i = \sum_{x \in G} n_S(x)\chi(g_i x) = \chi(g_i)\left(\sum_{x \in G} n_S(x)\chi(x)\right).$$

Clearly, by construction of $n_S(x)$ for $x \in G$, we can re-write $\sum_{x \in G} n_S(x)\chi(x)$ as simply $\sum_{s \in S} \chi(s)$ (where we adopt the convention that any element that appears with multiplicity m in S is counted m times in the sum). Thus we see

$$(Av_\chi)_i = \chi(g_i)\left(\sum_{s \in S} \chi(s)\right) = (v_\chi)_i\left(\sum_{s \in S} \chi(s)\right).$$

In other words,

$$Av_\chi = \left(\sum_{s \in S} \chi(s)\right) v_\chi,$$

and v_χ is by definition an eigenvector of A with eigenvalue $\left(\sum_{s \in S} \chi(s)\right)$. As there are $|G|$ distinct characters and they are all linearly independent by results in character theory (see [3]), we have $|G|$ independent eigenvectors v_χ and there can be no additional eigenvectors of the $|G| \times |G|$ matrix A. Thus $\left(\sum_{s \in S} \chi(s)\right)$ is an eigenvalue for every irreducible character χ, and there are no other eigenvalues, precisely as we wanted. □

Theorem 4.1 gives an easy way to compute the spectra of many k-regular graphs which can be constructed as Cayley graphs of finite abelian groups. We will now give several examples, and also use the opportunity to showcase some Ramanujan graphs.

Example 4.1. *The complete graph on $k+1$ vertices.*
The complete graph on $k+1$ vertices consists of $k+1$ vertices, where there is exactly one edge connecting each vertex to each other vertex. Thus it is a k-regular graph. We can construct the complete graph on $k+1$ vertices as a Cayley graph of the additive group

$G = \frac{\mathbb{Z}}{(k+1)\mathbb{Z}}$. In order to accomplish this construction, take $S = G - \{0\}$. Then the vertices of $X(G, S)$ are the $k+1$ elements of G, and a vertex x is connected to another vertex y if and only if $x - y \neq 0$, i.e. every vertex is connected to every other vertex. Since every element of S has multiplicity 1 in S, there is only one edge connecting x to y, and we see that $X(G, S)$ is precisely the complete graph on $k+1$ vertices.

To find the spectrum, we simply apply Theorem 4.1. Here every irreducible character is specified uniquely by its effect on the generator 1 of the cyclic group G. We see that the characters are χ_m for $m = 0, ..., k$, where

$$\chi_m(1) = e^{\frac{2\pi i m}{k+1}}.$$

Note that $\chi_0(x) = 1$ for every $x \in G$, i.e. χ_0 is the trivial character of G. Now, by Theorem 4.1, this gives corresponding eigenvalues $\lambda_0, ..., \lambda_k$, where

$$\lambda_m = \sum_{s \in S} \chi_m(s)$$

To evaluate this sum, recall that by results from character theory,

$$\sum_{g \in G} \chi(g) = 0$$

for any nontrivial irreducible character χ of G. Thus in particular, for $m \neq 0$,

$$\sum_{s \neq 0,\ s \in G} \chi_m(s) = 0 - \chi_m(0) = 0 - 1 = -1.$$

On the other hand, if $m = 0$, then

$$\sum_{s \neq 0,\ s \in G} \chi_m(s) = k.$$

Thus we see that the spectrum of $X(G, S)$ consists of k with multiplicity 1 (as anticipated by our previous results on eigenvalues of k-regular graphs in Section III) and -1 with multiplicity k.

In addition, as $1 = |-1| \leq 2\sqrt{k-1}$ for $k \geq 2$, we see that the complete graph on $k+1$

vertices is, in fact, Ramanujan whenever it has three or more vertices.

Example 4.2. *The cyclic graph on n vertices.*

The cyclic graph on n vertices consists of n vertices $v_1, ..., v_n$, where v_i is connected to v_{i+1} by a single edge for each i, v_n is connected to v_1 by a single edge, and there are no other edges. Thus the graph is clearly 2-regular. To construct this graph as a Cayley graph of an abelian group, again we will take our group to be a quotient of $\mathbb{Z}$ as in the previous example. Here let $G = \frac{\mathbb{Z}}{n\mathbb{Z}}$, so that $|G| = n$. We take $S = \{1, -1\}$ to be our symmetric multiset. Then in the Cayley graph $X(G, S)$, two vertices i and j are connected if and only if $i - j = 1$ or -1. It follows that i is connected to $i + 1$ by exactly one edge for each i, $n - 1$ is connected to 0 by exactly one edge, and there are no other edges. Thus $X(G, S)$ is precisely the cyclic graph on n vertices. Again we apply Theorem 4.1 to find the spectrum of $X(G, S)$.

The characters are the same as in the previous example: $\chi_0, ..., \chi_{n-1}$, where

$$\chi_m(1) = e^{\frac{2\pi i m}{n}}.$$

The corresponding eigenvalues given by Theorem 4.1 are, for $m = 0, ..., n - 1$,

$$\lambda_m = \sum_{s \in S} \chi_m(s) = \sum_{s \in \{-1,1\}} \chi_m(s) = e^{\frac{-2\pi i m}{n}} + e^{\frac{2\pi i m}{n}},$$

or simply

$$\lambda_m = 2\cos\left(\frac{2\pi m}{n}\right).$$

In particular, we see

$$|\lambda_m| \leq 2,$$

for all m. As $2 = 2\sqrt{2-1}$, this shows that the graph is Ramanujan by definition.

Note also that in particular $|\lambda_m| = 2$ if and only if $\frac{2m}{n}$ is an integer, which (since $m \leq n$ for $m = 0, ..., n - 1$) occurs only if $m = n/2$ is an integer or if $m = 0$. If $m = 0$ we get $\lambda_m = 2$ (this corresponds to the trivial character, and gives the expected eigenvalue of 2 since the graph is 2-regular). However, the first case, $m = n/2$, only makes sense if n is even. In this case we find $\lambda_{n/2} = 2\cos(\pi) = -2$. Recall from Section III that for a k-regular graph, $-k$ is an eigenvalue if and only if the graph is bipartite. Thus in fact we see from our calculations that in particular, the cyclic group on n vertices is bipartite if and only if n is even (which we can easily verify by other arguments as well).

We will finally consider one more example of some interest.

Example 4.3.
Again take our group to be the cyclic group $G = \frac{\mathbb{Z}}{n\mathbb{Z}}$. We now let S be the group of multiplicative units in G. By results in elementary number theory, the elements of S are precisely those congruence classes modulo n that consist of elements relatively prime to n. In particular, if n is prime then this reduces to the case in Example 4.1 of the complete graph on n vertices. We denote the graph $X(G, S)$ constructed in this way by $U(n)$.
Again by elementary results we have that $|S| = \phi(n)$, where ϕ is the Euler totient function. Thus $X(G, S)$ is a $\phi(n)$-regular graph. The eigenvalues are easily computed once again via Theorem 4.1.
The characters $\chi_0, ..., \chi_{n-1}$ are identical to those in Example 4.2. This gives eigenvalues $\lambda_0, ..., \lambda_{n-1}$ of

$$\lambda_m = \sum_{s \in S} \chi_m(s) = \sum_{(s,n)=1} e^{\frac{2\pi i s m}{n}}.$$

We can also express λ_m as a sum over cosines using the usual expression for the cosine function in terms of a sum of complex exponential functions:

$$\lambda_m = \sum_{(s,n)=1} \cos\left(\frac{2\pi s m}{n}\right).$$

Let us consider, for example, $n = 8$. Then the elements of S are the congruence classes of integers relatively prime to 8. There are only four such congruence classes, with representatives 1, 3, 5, and 7. Thus $S = 1, 3, 5, 7$, and $X(G, S)$ is 4-regular. The eigenvalues are easily computed to be 4 and -4 with multiplicity of 1, and 0 with multiplicity of 6, via the formula above. Thus we see that the corresponding graph is Ramanujan and bipartite. On the other hand, if we take $n = 9$, then $S = 1, 2, 4, 5, 7, 8$, and similar computations find eigenvalues 6 with multiplicity 1, -3 with multiplicity 2, and 0 with multiplicity 6. Again the graph is Ramanujan, but this time it is not bipartite.

Now consider $n = 27$. The elements of S are the congruence classes of integers relatively prime to 27, i.e. to 3. These are 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, and 26. Using the cosine formula above, -9 is calculated to be an eigenvalue of multiplicity 2 (corresponding to m=9 and m=18). However, here $\phi(27) = 27(1 - 1/3) = 18$, so the graph is 18-regular. We see that $2\sqrt{18-1} = 2\sqrt{17} \approx 8.246 < 9$. Thus in this case, the graph is *not* Ramanujan. This shows that in fact there are examples of abelian Cayley graphs which are not Ramanujan. Later in this section we will show that in fact, there are only finitely many k-regular Ramanujan abelian Cayley graphs for all $k > 2$.

For arbitrary n, we note that the eigenvalues,

$$\lambda_m = \sum_{s \in S} \chi_m(s) = \sum_{(s,n)=1} e^{\frac{2\pi i s m}{n}},$$

are in the form of Ramanujan sums (see [13]). In fact, there is a closed form evaluation of sums of this form in terms of the Euler totient function ϕ and the Mobius function μ. In this case, we get

$$\lambda_m = \mu\left(\frac{n}{(m,n)}\right) \frac{\phi(n)}{\phi\left(\frac{n}{(m,n)}\right)}.$$

Here the Mobius function is 1 if its argument is square-free with an even number of distinct prime factors, -1 if its argument is square-free with an odd number of distinct prime factors, and 0 if its argument is not square-free. The Euler totient function is defined as usual.

We note that if n is even, then an easy computation shows that $m = n/2$ gives an eigenvalue of $\lambda_m = -\phi(n)$. As the graph is $\phi(n)$-regular, this shows that the graph is bipartite whenever n is even, by Proposition 2.3.

Now we consider the question of which graphs of this type are Ramanujan. Let n be represented by its prime power decompositon, i.e. $n = 2^a p_1^{k_1} ... p_r^{k_r}$ for distinct primes $p_1 < p_2 < ... < p_r$, a a nonnegative integer, and positive integers $k_1, ..., k_r$. In order for the graph $U(n)$ to be Ramanujan, we need

$$\lambda(U(n)) \leq 2\sqrt{\phi(n) - 1}.$$

Here, $\lambda(U(n))$ is the maximal absolute value of the eigenvalues λ_m, smaller than $\phi(n)$. This is obtained by maximizing the value of

$$\frac{\phi(n)}{\phi\left(\frac{n}{(n,m)}\right)}$$

for $1 \leq m \leq n$ such that $\frac{n}{(n,m)}$ is square-free. In order for this squarefree condition to hold, we must have $2^{a-1}p_1^{k_1-1}\cdots p_r^{k_r-1}|m$. Now, in order to maximize the expression above, we need $\phi\left(\frac{n}{(n,m)}\right)$ to be minimized, but still be greater than 1 (since otherwise the eigenvalue has absolute value $\phi(n)$, which we therefore ignore).

Here, it is easy to see that $\phi\left(\frac{n}{(n,m)}\right) = (q_1-1)...(q_t-1)$ where $q_1, ..., q_r$ are the prime factors of n that have precisely one less power in the prime factorization of m than in the prime factorization of n. If $r \geq 1$, then the desired minimization is achieved by taking $m = n/p_1$, to get $\phi\left(\frac{n}{(n,m)}\right) = p_1 - 1$.

We now consider several cases. First, suppose $r = 0$, so that $n = 2^a$ for some integer $a > 0$. Then the only possibilities for m giving a nonzero eigenvalue are $m = 2^{a-1}$ or $m = 2^a = n$. In the first case, the eigenvalue is $-\phi(n)$ and in the second case the eigenvalue is $\phi(n)$. This shows that if $n = 2^a$, then the graph $U(n)$ is certainly Ramanujan.

Next, suppose $r = 1$, $a = 0$. We have already considered $n = p$ for some prime $p > 2$, and concluded that the graph is Ramanujan. In the case $n = p^2$ for a prime $p > 2$, we minimize $\phi\left(\frac{n}{(n,m)}\right)$ by taking $m = p$, as discussed above. We can then compute the corresponding eigenvalue, to get absolute value $\frac{\phi(n)}{p-1} = \frac{p(p-1)}{p-1} = p$, which is no larger than $\sqrt{p(p-1)-1}$ for every prime $p \geq 3$. Thus we conclude that $U(p^2)$ is Ramanujan for all primes p. Finally, consider the case $n = p^k$ for $p, k > 2$. In this case, we maximize the eigenvalue by taking $m = p^{k-1}$, and its absolute value is $\frac{p^{k-1}(p-1)}{p-1} = p^{k-1}$. Now, we find that $p^{k-1} > 2\sqrt{p^{k-1}(p-1)-1}$ for all primes $p, k \geq 3$, and so $U(p^k)$ is not Ramanujan for any prime $p \geq 3$, $k \geq 3$.

Next, suppose n has two prime factors, so $n = p^a q^b$ for some primes p, q with $p < q$. First assume $a > 1$ or $b > 1$. If $p > 2$, then $q \geq 5$ and the maximal absolute value of an eigenvalue, smaller than $\phi(n)$, is achieved by taking $m = n/p = p^{a-1}q^b$. The eigenvalue then has magnitude $\frac{\phi(n)}{p-1}$. Note that

$$\phi(n) = p^{a-1}q^{b-1}(p-1)(q-1) > 4(p-1)^2,$$

since $q-1 \geq 4$ and $p^{a-1}q^{b-1} > p-1$. It follows by simple manipulation that in this case, $\frac{\phi(n)}{q-1} \geq 2\sqrt{\phi(n)} > 2\sqrt{\phi(n)-1}$, meaning that the graph $U(n)$ is not Ramanujan. On the other hand, if $p=2$, i.e. n is even, then this argument breaks down, and we see that the largest absolute value of an eigenvalue, smaller than $\phi(n)$, is achieved by taking $m = n/q$ instead. In this case, the eigenvalue has magnitude $\frac{\phi(n)}{q-1}$. If $a \geq 3$, $b > 1$, then we can easily see

$$\frac{\phi(n)}{q-1} = q^{b-1}2^{a-1}(2-1) = q^{b-1}2^{a-1} > 4(q-1),$$

(since then $2^{a-1} \geq 4$ and $q^{b-1} > q-1$) which can be rearranged to show that the Ramanujan condition is violated as above in the $p > 2$ case. However, if $a \leq 2$ or $b = 1$, it is possible for the graph to be Ramanujan. For example, it is easy to see that $U(124)$ is Ramanujan (note $124 = 2^2 \times 31$, i.e. $a = 2$ and $b = 1$) is Ramanujan, as are $U(248)$ (where $248 = 2^3 \times 31$, i.e. $a > 2$ and $b = 1$) and $U(18)$ (where $18 = 2 \times 3^2$, i.e. $a \leq 2$ and $b > 1$).

Finally, there is the case that $a = b = 1$ for arbitrary p, q. In this case it is again possible that the resulting graphs $U(n)$ will be Ramanujan. For example, it is easy to see that $U(15)$ and $U(899)$ are Ramanujan. In general, if $a = b = 1$ and q is sufficiently close to p, we will have a Ramanujan graph. More detailed analysis of these remaining cases will be saved for a paper to follow.

We now move on to considering more than two factors, i.e. the general case with $n = 2^a p_1^{k_1}...p_r^{k_r}$, $p_1 < p_2 < ... < p_r$. First consider n odd, so that $a = 0$ and $n = p_1^{k_1}...p_r^{k_r}$, and assume $r \geq 3$ (i.e. at least three prime factors, since we have already considered the cases of one or two prime factors above). As discussed previously, the maximal absolute value of an eigenvalue, not equal to $\phi(n)$, is achieved by taking $m = n/p_1$. The eigenvalue itself is then $\phi(n)/(p_1-1)$. In this case, it is easy to see that

$$\phi(n) \geq (p_1-1)(p_2-1)...(p_r-1) > 4(p_1-1)^2,$$

since $p_1 \geq 3$ implies $p_2 \geq 4$ and $p_3 - 1 > p_1 - 1$. As in the case of two prime factors, this can easily be rearranged to show that the Ramanujan condition is violated, and thus if n is odd and has at least three distinct prime factors, then $U(n)$ is not Ramanujan.

Lastly, suppose n is even. If $r \geq 3$, then the argument above for the odd case can be applied nearly identically to show that $U(n)$ is not Ramanujan. This permits the general statement that if n has at least three distinct odd prime factors, then $U(n)$ is not Ramanujan. On the other hand, if $r = 2$, then $n = 2^a p^b q^c$ for some distinct odd primes $p < q$ and integers $a, b, c \geq 1$. In this case the maximal absolute value of an eigenvalue smaller than $\phi(n)$ is achieved by letting $m = n/p$, to obtain an eigenvalue of magnitude $\phi(n)/(p-1)$. If $a > 2$ and $b > 1$ or $c > 1$, we easily see that the graph is not Ramanujan by the same argument as in the even case with two distinct prime factors considered above, since then it is clear that

$$\phi(n) = 2^{a-1} p^{b-1} q^{c-1} (p-1)(q-1) > 4(p-1)^2.$$

In fact, in order for the inequality above not to hold, we would in particular need that $2^{a-1} p^{b-1} q^{c-1} \leq 4$. Thus certainly we must in any case have that $a < 4$, and either $b = c = 1$ or $p = 3, b = 2, a = c = 1$, if the graph $U(n)$ is to be Ramanujan. In fact, we find that some graphs of this type are Ramanujan. For example, it is easy to verify that $U(7198)$ is Ramanujan (where $7198 = 2 \times 59 \times 61$, i.e. $a = b = c = 1$). More detailed discussion of these cases will also be saved for a future paper.

We now show that, while we can get examples of Ramanujan graphs via Cayley graphs of abelian groups as demonstrated in the previous three examples, we cannot construct infinite families of Ramanujan graphs in this way (except in a fairly trivial case). The argument is based on a theorem of Dirichlet, which follows from the pigeonhole principle. We will first state this theorem and then present the proof on nonexistence of such families. This result was originally proved by Murty [12]. There is a much more detailed treatment with improved results in [4].

Theorem 4.2. Let $s_1, ..., s_k$ be arbitrary positive real numbers, and let $N > 1$ be an integer. Then there exists $m \in \mathbb{Z}$, $0 < m < N^k$ and $m_1, ..., m_k \in \mathbb{Z}$ such that

$$|ms_j - m_j| < \frac{2}{N}$$

for all j, $1 \leq j \leq k$.

Roughly speaking, this says that there is some integer m such that ms_j is within $\frac{2}{N}$ of some integer for each j, regardless of the numbers s_j or N. We also recall the following structure theorem for finite abelian groups, which will be required in our proof on infinite families of Cayley graphs of Abelian groups. For a proof of this theorem, see [3].

Theorem 4.3.
Let R be a Principal Ideal Domain, and let M be a finitely generated R-module. Then M is isomorphic to a direct sum of finitely many cyclic R-modules, i.e.

$$M \cong R^r \oplus \frac{R}{(a_1)} \oplus ... \oplus \frac{R}{(a_k)},$$

where $a_1, ..., a_k \in R$ and $a_1 | a_2 | ... | a_k$.

Noting that every abelian group is a $\mathbb{Z}$-module and combining with the fact that $\mathbb{Z}$ is a Principal Ideal Domain with an infinite number of elements, we have the following result for finite abelian groups.

Corollary 4.1.
Let G be a finite abelian group. Then

$$G \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \oplus ... \oplus \frac{\mathbb{Z}}{n_k\mathbb{Z}},$$

where k is a non-negative integer and $n_1, ..., n_k \in \mathbb{Z}$ with $n_1|n_2|...|n_k$.

We are now prepared to prove our theorem on infinite families of Cayley graphs of abelian groups.

Theorem 4.4.
Let G be a finite abelian group and S be a symmetric subset of G, with $|S| = k$, and assume $X(G, S)$ is connected. Then the second largest eigenvalue of the Cayley graph $X(G, S)$ is $k - o(1)$ as $|G| \to \infty$. This in particular shows that $\lambda(X) \leq 2\sqrt{k-1}$ for only finitely many such pairs (G, S) as long as $2\sqrt{k-1} < k$, i.e. as long as $k > 2$.
Proof: We will first consider the simple case $G = \frac{\mathbb{Z}}{n\mathbb{Z}}$ for some $n \in \mathbb{N}$, and use the above results on the structure of finite abelian groups to extend this case to all finite abelian

groups. Assume $S = a_1, ..., a_k$ and $X = X(G, S)$. Let $s_j = a_j/n$ for each j, $1 \leq j \leq k$, and let $N = \left[n^{1/k}\right]$. Note that the maps from G to $\mathbb{C}$ given by

$$g \mapsto e^{2\pi i m g/n}$$

are nontrivial characters of G for $1 \leq m \leq n-1$. Thus by Theorem 4.1, we have nontrivial eigenvalues of X given by

$$\sum_{j=1}^{k} e^{2\pi i m a_j/n} = \sum_{j=1}^{k} e^{2\pi i m s_j}$$

corresponding to these characters. In fact, because S is symmetric, this can necessarily be rewritten as

$$\sum_{j=1}^{k} \cos(2\pi m s_j),$$

using the usual exponential representation of the cosine function.

Now, by Dirichlet's result (Theorem 4.2), there is some such m, $0 < m < N^k \leq n$ and some integers $m_1, ..., m_k$ such that

$$|m s_j - m_j| < 2/N$$

for each j, $1 \leq j \leq k$. Recalling the Taylor series for the cosine function,

$$\cos x = 1 - \frac{x^2}{2} + ...,$$

this shows that for sufficiently large n (i.e. sufficiently large $|G|$, and correspondingly small N), we have precisely that the eigenvalue in question is $k - O(k/N^2)$, since each term in the sum is $1 - O(1/N^2)$. Thus in particular, for sufficiently large n, the largest nontrivial eigenvalue is larger than $2\sqrt{k-1}$ unless $k = 2\sqrt{k-1}$, i.e. unless $k = 2$. So there are no infinite families of k-regular Ramanujan abelian Cayley graphs of groups of the form

$$\frac{\mathbb{Z}}{n\mathbb{Z}}$$

unless $k = 2$. This completes the proof in the case that $G = \frac{\mathbb{Z}}{n\mathbb{Z}}$.

We are now ready to look at the general case. Here first note that S generates G because the Cayley graph is connected (connectedness implies that there is a path from the identity element of G to every other element, corresponding to additions by elements in S, and thus shows that S must be a generating set). By Corollary 4.1,

$$G \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \oplus ... \oplus \frac{\mathbb{Z}}{n_t\mathbb{Z}}$$

for some $n_1|n_2|...|n_t$. In particular, n_t is the smallest number such that $g^{n_t} = 1$ for all $g \in G$. As $|G| \to \infty$, we must necessarily have $n_t \to \infty$ as well, since otherwise the abelian group G generated by the k-set S would have bounded size, a contradiction.

Thinking of G as the direct sum above, there is a projection map $\omega : G \to \frac{\mathbb{Z}}{n_t\mathbb{Z}}$. Again, assume $S = a_1, ..., a_k$ and $X = X(G, S)$. Let $s_j = \omega(a_j)/n_t$ for each j, $1 \leq j \leq k$, and let $N = \left\lfloor n_t^{1/k} \right\rfloor$. Similar to the first case, the map

$$g \mapsto e^{2\pi i m\omega(g)/n_t}$$

is a nontrivial character of G for each m, $1 \leq m \leq n-1$. The eigenvalues corresponding to these characters are again given by Theorem 4.1 as

$$\sum_{j=1}^{k} e^{2\pi i m\omega(a_j)/n_t} = \sum_{j=1}^{k} e^{2\pi i m s_j}.$$

The rest of the argument is identical to that used in the simple case above to see that for some m, this eigenvalue is $k - O(k/n_t^{2/k})$. As noted above, $n_t \to \infty$ as $|G| \to \infty$ and thus as $|G| \to \infty$, the second largest eigenvalue is at least $k - o(1)$, as we wanted. □

This proof showed that there are no infinite families of Ramanujan abelian Cayley graphs in every case except one: If $k = 2$, then there is no contradiction, since $2 = 2\sqrt{2-1}$. For example, all of the cyclic graphs are 2-regular abelian Cayley graphs and are also Ramanujan as per Example 4.2 above.

VI. THE ALON-BOPPANA THEOREM

In this section we will prove the results of Alon-Boppana and Nilli stated in Section IV regarding bounds on the second largest eigenvalue. These results provide significant motivation for the study of Ramanujan graphs. The proof is correspondingly more involved than for the simple results presented earlier.

We will begin by restating the theorems for reference.

Theorem 3.2.

$$\mathrm{liminf}_{\mathrm{n}\to\infty}\lambda(X_{n,k}) \geq 2\sqrt{k-1},$$

where the lim inf is taken over k-regular graphs with $|V(X_{n,k})| = n \to \infty$.

Theorem 3.3. If $\mathrm{diam}(X) \geq 2b+2 \geq 4$ for some natural b, then

$$\lambda(X) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

As shown earlier (Proposition 3.2), Theorem 3.2 follows from Theorem 3.3. Before we begin the proof, we will give some results from linear algebra and use them to develop our strategy. We recall the Rayleigh-Ritz theorem.

Theorem 5.1. Let A be a symmetric matrix with maximal and minimal eigenvalues λ_M and λ_m, respectively. Then

$$\lambda_M = \max_{v\neq 0} \frac{(Av,v)}{(v,v)}$$

and

$$\lambda_m = \min_{v\neq 0} \frac{(Av,v)}{(v,v)}.$$

Proof: As A is symmetric, it is orthogonally diagonalized by some orthogonal matrix U, i.e.

$$A = UDU^t$$

for some diagonal matrix D whose diagonal entries are the eigenvalues of A. In particular, for any vector v,

$$(Av, v) = v^t Av = v^t UDU^t v = \sum_i \lambda_i |(U^t v)_i|^2$$

where λ_i are the eigenvalues of A. This certainly implies

$$\lambda_m \sum_i |(U^t v)_i|^2 \leq (Av, v) \leq \lambda_M \sum_i |(U^t v)_i|^2.$$

Here U is orthogonal and thus preserves magnitudes. In particular, $\sum_i |(U^t v)_i|^2 = ||v||^2 = v^t v = (v, v)$. Then as long as $v \neq 0$,

$$\lambda_m(v, v) \leq (Av, v) \leq \lambda_M(v, v),$$

from which the Rayleigh-Ritz theorem follows since equality is achieved on the respective sides when we consider the eigenvectors corresponding to the eigenvalues λ_m and λ_M. $\square$

Now let X be a simple k-regular graph with n vertices and let $L(X)$ denote the space of real functions on X. $L(X)$ has a natural inner product defined by

$$(f, g) = \sum_{x \in X} f(x)g(x)$$

for all $f, g \in L(X)$. Matrices act on $L(X)$ in the usual way. In particular, if A is the adjacency matrix for X, then

$$(Af)(x) = \sum_{(x,y) \in E(X)} f(y)$$

(i.e. the sum is over all vertices y adjacent to x. This is the same as multiplying A with a column vector whose entry in the ith position is the value of f at vertex i). Here k is an eigenvalue corresponding to the set of constant functions. As X is simple, Proposition 2.2 shows that there are no other functions with k as an eigenvalue,. It is clear that the space of constant functions is the space spanned by the function f_0 such that $f_0(x) = 1$ for all

$x \in X$. We can write

$$L(X) = \mathbb{R}f_0 \oplus L_p(X)$$

where $L_p(X)$ is the orthogonal space to f_0. Considering A acting only on $L_p(X)$, the Rayleigh-Ritz theorem gives

$$\lambda_1(X) = \max_{(f,f_0)=0, f\neq 0} \frac{(Af,f)}{(f,f)}$$

where λ_1 is the second largest eigenvalue of A on the whole space $L(X)$ as defined earlier (we know that the largest eigenvalue is k, corresponding to the direct summand $\mathbb{R}f_0$ which we have ignored). In slightly different notation, we can write $\Delta = kI - A$ where I is the identity matrix, with eigenvalues $k - \lambda_i$, and we get

$$k - \lambda_1(X) = \min_{(f_0,f)=0, f\neq 0} \frac{(\Delta f,f)}{(f,f)}.$$

This will be our starting point for the proof of Theorem 3.3 (and thus Theorem 3.2).

Proof: (Of Theorem 3.3). As in the theorem, assume $\mathrm{diam}(X) \geq 2b+2$, and let u, v in X such that $d(u,v) \geq 2b+2$. Then define for each $i \geq 0$

$$U_i = \{x \in X | d(x,u) = i\}$$

and

$$V_i = \{x \in X | d(x,v) = i\}.$$

Then $U_0, ..., U_b, V_0, ..., V_b$ are necessarily distinct sets, since the triangle inequality would otherwise imply a contradiction ($d(u,v) \leq d(u,x) + d(x,v) \leq 2b$ if x is in one of these U_i's and also one of the V_j's). Furthermore, no vertex in any of $U_0, ..., U_b$ is adjacent to any vertex in $V_0, ..., V_b$ for the same reason (we would then have $d(u,v) \leq 2b+1$, another contradiction). As X is k-regular and connected, by definition each vertex in U_i is adjacent to at least one vertex in U_{i-1} and at most $k-1$ vertices in U_{i+1}, i.e.

$$|U_{i+1}| \leq (k-1)|U_i| = q|U_i|$$

if $q = k - 1$, and similarly

$$|V_{i+1}| \leq q|V_i|$$

(for more on the topic of matrix analysis of this type, see [6]).

It follows that $|U_b| \leq q^{b-i}|U_i|$ for $b \geq i \geq 1$ (and similarly for the V_i's).

Now we define $f \in L(X)$ by $f(x) = F_i$ for $x \in U_i$ and $f(x) = G_i$ for $x \in V_i$, with F_i and G_i to be specified below, and $f(x) = 0$ otherwise. Here by definition

$$(f, f) = \sum_{x \in X} f(x)^2 = A + B$$

where

$$A = \sum_{i=0}^{b} F_i^2 |U_i|, \ \ B = \sum_{i=0}^{b} G_i^2 |V_i|.$$

We see that we can choose $F_0 = \alpha, G_0 = \beta, F_i = \alpha q^{-(i-1)/2}, G_i = \beta q^{-(i-1)/2}$ for each i, where α and β are selected to force $(f, f_0) = 0$.

We note the following identity which we will require below:

$$\frac{1}{2} \sum_{(x,y) \in E(X)} (f(x) - f(y))^2 = k(f, f) - (Af, f) = (\Delta f, f),$$

where the sum is taken over all ordered pairs of vertices (x, y) such that $(x, y) \in E(X)$. To see this, simply expand the square and use the definitions, to get

$$\frac{1}{2} \sum_{(x,y) \in E(X)} (f(x) - f(y))^2 = \frac{1}{2} \left(\sum_{(x,y) \in E(X)} f(x)^2 + \sum_{(x,y) \in E(X)} f(y)^2 \right) - \sum_{(x,y) \in E(X)} f(x)f(y).$$

The two terms in the parentheses are each $k(f, f)$ by definition, and the last term can be rewritten as $\sum_{y \in X} \left(\sum_{(x,y) \in E(X)} f(x) \right) f(y)$, which is precisely (Af, f).

Now let $U = \bigcup_{i=1}^{b} U_i$, $V = \bigcup_{i=1}^{b} V_i$. By previous arguments, U and V are disjoint and no vertex of V is adjacent to a vertex of U, and f is by definition non-zero only on $U \cup V$. By our note above, we can thus write

$$(\Delta f, f) = \frac{1}{2} \sum_{(x,y) \in E} (f(x) - f(y))^2 = A_V + A_U$$

where

$$A_V = \frac{1}{2} \sum_{(x,y)\in E,\ x \text{ or } y\in V} (f(x) - f(y))^2$$

and

$$A_U = \frac{1}{2} \sum_{(x,y)\in E,\ x \text{ or } y\in U} (f(x) - f(y))^2.$$

We can give estimates for these by recalling that X is k-regular and separating by contributions from each V_i and U_i. In fact, we can see that

$$A_U \leq \sum_{i=1}^{b-1} |U_i| q (q^{-(i-1)/2} - q^{-i/2})^2 \alpha^2 + |U_b| q q^{-(b-1)} \alpha^2.$$

Here the last term arises from elements of U_b, recalling that f is 0 on all vertices adjacent to U_b that are not in U. Each term of the sum corresponds to elements of U_i for the corresponding i, simply plugging in for f and using the upper bound of $k-1$ elements in U_{i+1} for each element in U_i. Similarly,

$$A_V \leq \sum_{i=1}^{b-1} |V_i| q (q^{-(i-1)/2} - q^{-i/2})^2 \beta^2 + |V_b| q q^{-(b-1)} \alpha^2.$$

Expanding the estimate for A_U and simplifying terms, we obtain

$$A_U \leq (\sqrt{q} - 1)^2 \left(\sum_{i=1}^{b} |U_i| q^{-(i-1)} \right) \alpha^2 + \alpha^2 (2\sqrt{(q)} - 1) |U_b| q^{-(b-1)}.$$

By definition of A, this is precisely

$$A_U \leq (\sqrt{q} - 1) \left(A - \alpha^2 |U_0| \right) + (2\sqrt{q} - 1) \frac{A - \alpha^2}{b},$$

where the last term arises because $q^{-(i-1)}|U_i| \geq q^{-i}|U_{i+1}|$, i.e. $|U_b| q^{-(b-1)} \leq q^{-(k-1)} |U_k|$ for each k, and $A = \sum_{i=0}^{b} |U_i| \alpha^2 q^{-(i-1)}$.

We can ignore the negative terms involving $-\alpha^2$ to obtain the strict inequality

$$A_U < A \left(1 + q - 2\sqrt{(q)} + \frac{2\sqrt{q} - 1}{b} \right).$$

By identical calculations for A_V we would have

$$A_V < B\left(1 + q - 2\sqrt{q} + \frac{2\sqrt{q} - 1}{b}\right).$$

Since we had

$$(\Delta f, f) = A_U + A_V,$$

this gives

$$(\Delta f, f) \leq (A + B)\left(1 + q - 2\sqrt{(q)} + \frac{2\sqrt{q} - 1}{b}\right).$$

Here we had $(f, f) = A + B$, so

$$\frac{(\Delta f, f)}{(f, f)} \leq \left(1 + q - 2\sqrt{(q)} + \frac{2\sqrt{q} - 1}{b}\right),$$

and the Rayleigh-Ritz theorem gives

$$k - \lambda_1(X) = \min_{(f, f_0) = 0, f \neq 0} \frac{(\Delta f, f)}{(f, f)} < \left(1 + q - 2\sqrt{(q)} + \frac{2\sqrt{q} - 1}{b}\right).$$

Recalling that $q = k - 1$, this says precisely

$$\lambda_1(X) > 2\sqrt{k - 1} - \frac{2\sqrt{k - 1} - 1}{b},$$

which is Nilli's result stated above and concludes the proof. □

As we proved before (in Section IV) and have mentioned several times, this result also implies the Allon-Boppana theorem.

VII. RAMANUJAN GRAPHS AS EXPANDERS

Let X be a k-regular graph with n vertices. We start with some definitions.

Definition 6.1. The *boundary* of a subset A of $V(X)$ is denoted by ∂A, with the natural definition

$$\partial A = \{x \in V(X) | d(x, A) = 1\}.$$

In other words, the boundary of a subset is the set of vertices not in the subset but which are adjacent to some vertex in the subset.

This allows us to define the notion of an expander graph. Let $V = V(X)$.

Definition 6.1. X is referred to as a c-expander (for a positive real number c) if

$$\frac{|\partial A|}{|A|} \geq c$$

for all subsets A of V with $|A| \leq |V|/2$.

We will relate the theory of c-expanders to that of the eigenvalues of a graph, and show that Ramanujan graphs in particular have an interesting property as expanders.

Given a subset A of V, define a real function f_A on the vertices of X by $f_A(x) = |V \backslash A|$ if $x \in A$ and $f_A(x) = -|A|$ otherwise. We use the theory discussed in the last section to examine f. For $f = f_A$, the inner product (f, f_0) is

$$(f, f_0) = \sum_{x \in X} f(x) f_0(x) = \sum_{x \in X} f(x) = |V \backslash A| \sum_{x \in A} 1 - |A| \sum_{x \in X/A} 1 = |V \backslash A||A| - |A||V \backslash A| = 0,$$

so f is orthogonal to the constant function. By the discussion in the previous section, this means

$$k - \lambda_1(X) \leq \frac{(\Delta f, f)}{(f, f)}.$$

Now, we easily see that

$$\begin{aligned} (f, f) = & \sum_{x \in X} f(x)^2 \qquad (4) \\ = & \sum_{x \in A} |V \backslash A|^2 + \sum_{x \in X/A} |A|^2 \end{aligned}$$

$$
\begin{aligned}
&= |A||V\backslash A|^2 + |V\backslash A||A|^2 \\
&= (|A| + |V\backslash A|)(|A||V\backslash A|) \\
&= |V||A||V\backslash A|.
\end{aligned}
$$

To find $(\Delta f, f)$, we use the identity

$$(\Delta f, f) = \frac{1}{2} \sum_{(x,y)\in E(X)} (f(x) - f(y))^2$$

proved in the last section. The only terms that contribute are those where $x \in A$ and $y \in V\backslash A$ or visa-versa, since f is fixed on A and its complement. Thus

$$(\Delta f, f) = \frac{1}{2}\left(2 \sum_{x\in A, y\in V\backslash A, (x,y)\in E} (|V\backslash A| + |A|)^2\right) = |V|^2 \sum_{x\in A, y\in V\backslash A, (x,y)\in E} 1 = |V|^2|\partial A|.$$

The Rayleigh-Ritz theorem then gives

$$\frac{|V||\partial A|}{|A||V\backslash A|} = \frac{|V|^2|\partial A|}{|V||A||V\backslash A|} = \frac{(\Delta f, f)}{(f, f)} \geq k - \lambda_1(X),$$

or

$$\frac{|\partial A|}{|A|} \geq (k - \lambda_1(X))\frac{|V\backslash A|}{|V|}.$$

When we are determining whether X is an expander, by definition $|A| \leq |V|/2$, so $|V\backslash A|/|V| \geq 1/2$. Thus here

$$\frac{|\partial A|}{|A|} \geq \frac{k - \lambda_1(X)}{2},$$

and X is a $(k - \lambda_1)/2$-expander. Ramanujan graphs are those for which λ_1 is minimized as their order goes to infinity (by the Alon-Boppana theorem, for example). Here, the expander constant is maximized for minimal $\lambda_1(X)$, so in fact we see that families of Ramanujan graphs are also families of good expanders (in the sense that they are c-expanders for large expander constants c).

The immediate significance of this fact is not obvious; however, expanders are of considerable significance in the theory of computer networks and computer science in general. Thus, this interesting property provides some additional practical motivation for the study of Ramanujan graphs. For more discussion of expanders, see [2], [17].

VIII. THE ZETA FUNCTION OF A GRAPH

In the last section, we will give the definition of the zeta function of a graph along with some simple results and motivation.

Recall that if A is the adjacency matrix for a k-regular graph X, then the (x, y)th entry of A^r is the number of walks of length r from x to y. We make a slightly different definition below.

Definition 6.1. A *proper walk* in a k-regular graph X is a walk with no immediate back-tracking (note this is not the same as a path, since we do not forbid repeated vertices, only patterns of the form u, v, u in the walk for any vertices u and v). The matrix whose (x, y)th entry is the number of proper walks of length r from x to y is denoted by $A_r(X)$.

For the remainder of this section, let X be a k-regular graph and $A_r = A_r(X)$ for each r.

Now, we can easily observe a simple recursion relation for A_r. Note that clearly $A_0 = I$, the identity matrix, since there is one walk of length 0 from each vertex to itself and no other walks of length 0. Similarly, $A_1 = A$, since A already encodes proper walks of length 1, and in fact,

$$A^2 = A_2 + kI,$$

since A^2 encodes walks of length 2 on the left, and the only way to obtain a non-proper walk of length 2 is to leave a vertex for any of the k adjacent vertices and then return, which is encoded by kI. The following result arises in a similar way.

Proposition 7.1. For $r \geq 2$

$$AA_r = A_{r+1} + (k-1)A_{r-1}.$$

Proof: The multiplication on the left side encodes walks of length $r+1$ that are extended from proper walks of length r. The addition on the right side does the same in a different manner. The first term encodes the number of proper walks of length $r+1$, and the second term encodes the number of ways to obtain non-proper walks of length $r+1$ that are extended from proper walks of length $r-1$ by leaving along one edge then backtracking along the same edge. □

Using the theory of recursion relations and formal power series, this allows us to derive the following relation.

Proposition 7.2.

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)(I - At + (k-1)t^2 I) = (1-t^2)I.$$

Proof: We simply multiply the power series on the left hand side and use Proposition 7.1. Here the coefficient of t^r in

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)(I - At + (k-1)t^2 I)$$

is, for $r \geq 2$,

$$A_r - AA_{r-1} + (k-1)A_{r-2}.$$

By Proposition 7.1, for $r > 2$ we have $(k-1)A_{r-2} = AA_{r-1} - A_r$. Thus we see that for $r > 2$, the coefficient of t^r is precisely 0. On the other hand, the coefficient of t^0 is just I. The coefficient of t is $A - A = 0$. Finally, the coefficient of t^2 is $A_2 - A^2 + (k-1)I$, and by the argument preceding Proposition 7.1, $A_2 - A^2 = -kI$. Thus the coefficient of t^2 is $-kI + (k-1)I = -I$.

These calculations together show precisely that

$$\left(\sum_{r=0}^{\infty} A_r t^r\right)(I - At + (k-1)t^2 I) = I - It^2 = (1-t^2)I,$$

as the proposition claims. □

The significance of this result will be explained later in this section. We now make some additional definitions in preparation for defining the zeta function of a graph.

Motivated by the theory of the Selberg zeta function, Ihara [7] defines zeta functions for k-regular graphs in an analogous manner. We will briefly define the zeta function of a graph here and state the main result in the theory of these zeta functions of graphs, as well as giving a small amount of discussion.

Definition 7.2. A *closed geodesic* in the k-regular graph X is a proper walk whose endpoints are equal. If γ is a closed geodesic, then γ^r is the closed geodesic corresponding to r repetitions of γ. A *prime* geodesic is a closed geodesic which is not a power of any other closed geodesic.

There is an obvious equivalence relation on closed geodesics in a graph which we will now define.

Definition 7.3. Two closed geodesics $y = (y_0, ..., y_n)$ and $x = (x_0, ..., x_m)$ are equivalent if and only if $m = n$ and for some nonnegative integer d, $y_i \equiv x_{i+d \bmod n}$ for all i. The equivalence classes of closed geodesics of a graph are referred to as *prime geodesic cycles*.

Given these definitions, we can give Ihara's definition of the zeta function of a graph and state the main result on these zeta functions.

Definition 7.3. The zeta function of the k-regular graph X is defined by

$$\zeta_X(s) = \prod_p \left(1 - (k-1)^{-s \text{ length}(p)}\right)^{-1},$$

where the product is over all prime geodesic cycles p (i.e. equivalence classes of closed geodesics, as defined above). Note in particular that the length of a prime geodesic cycle is certainly well-defined as the length of any member of the equivalence class, from the definition.

The following result has been proved by Ihara [7]. A sketch of the proof is also given in [11].

Theorem 7.1. Let $g = \frac{(k-2)|V|}{2}$, and let $q = k-1$. Then

$$\zeta_X(s) = (1-u^2)^{-g}\det(I - Au + qu^2 I)^{-1},$$

where $u = (k-1)^{-s}$. As well, all of the singular points s of $\zeta_X(s)$ with $0 < \mathrm{Re}(s) < 1$ have $\mathrm{Re}(s) = 1/2$ (i.e. $\zeta_X(s)$ satisfies the classical Riemann hypothesis) if and only if the k-regular graph X is a Ramanujan graph.

The proof of this result uses the encoding of the numbers of proper walks in the graph calculated above (in particular, the power series identity in Proposition 6.2).

In fact, zeta functions have been defined for arbitrary graphs with similar results [5], [15]. This topic illustrates some connections between graph theory and other topics; in particular, the subject of algebraic varieties. An interesting question is to what extent the properties of the zeta function of a graph correspond to properties of the graph itself.

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